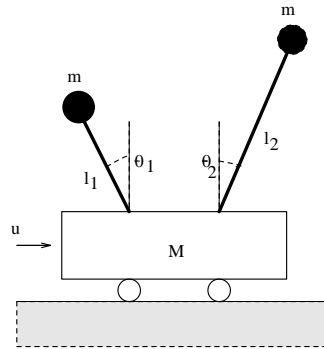
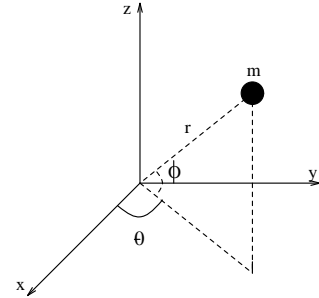


RLC Network



Cart with inverted penduli



Satellite

Dynamical Systems: Case Studies

1. Spring-mass system

A spring-mass system consisting of 3 masses m_1, m_2, m_3 , interconnected by 4 springs with constants k_1, k_2, k_3, k_4 , is described by the following set of differential equations:

$$\begin{aligned} m_1 \ddot{q}_1 &= -k_1 q_1 + k_2 (q_2 - q_1) \\ m_2 \ddot{q}_2 &= k_2 (q_1 - q_2) + k_3 (q_3 - q_2) \\ m_3 \ddot{q}_3 &= k_3 (q_2 - q_3) - k_4 q_3 \end{aligned}$$

where q_1, q_2, q_3 are the positions of the 3 masses (in the direction of movement).

We choose the state variables $x_1 = q_1, x_2 = \dot{q}_1, x_3 = q_2, x_4 = \dot{q}_2, x_5 = q_3, x_6 = \dot{q}_3$. The input u is a force applied to m_2 parallel to the direction of motion. The output is $y = x_5$.

2. RLC network

The RLC network above, with state variables as shown. The inputs u_1, u_2 are (independent) voltage sources. Outputs: $y_1 = x_2, y_2 = x_4$.

3. Orbiting satellite

$$x := \begin{pmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{pmatrix}, \quad \dot{x} = f(x, u) = \begin{pmatrix} \dot{r} \\ r\dot{\theta}^2 \cos^2 \phi + r\dot{\phi}^2 - k/r^2 + u_r/m \\ \dot{\theta} \\ -2\dot{r}\dot{\theta}/r + 2\dot{\theta}\dot{\phi} \sin \phi / \cos \phi + u_\theta/(mr \cos \phi) \\ \dot{\phi} \\ -\dot{\theta}^2 \cos \phi \sin \phi - 2\dot{r}\dot{\phi}/r + u_\phi/(mr) \end{pmatrix}$$

Output variables

$$y_1 = r, \quad y_2 = \theta, \quad y_3 = \phi$$

The following is an equilibrium solution of the above differential equations:

$$x_0(t) = [r_0 \quad 0 \quad \omega_0 t \quad \omega_0 \quad 0 \quad 0]^T$$

for $u = 0$ and $\omega_0^2 r_0^3 = k$ (gravitational constant). The linearized equations around this equilibrium solution are $\dot{x} = Ax + Bu$, where:

$$A = \left(\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2\omega_0 r_0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2\omega_0/r_0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_0^2 & 0 \end{array} \right), \quad B = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1/m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/mr_0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 1/mr_0 \end{array} \right)$$

4. Hot-air balloon

$$\begin{aligned} \dot{\theta} &= -\frac{1}{\tau_1}\theta + u & \theta &: \text{temperature change from equilibrium} \\ \dot{v} &= -\frac{1}{\tau_2}v + \sigma\theta + \frac{1}{\tau_2}w & v &: \text{vertical velocity} \\ \dot{h} &= v & u &: \text{proportional to heat added to air in balloon (control)} \\ y &= h & w &: \text{vertical wind velocity (disturbance input)} \\ & & h &: \text{change in altitude from equilibrium} \end{aligned}$$

5. Cart with inverted penduli

For small $|\theta_i|$ the equations of motion are:

$$\begin{aligned} M\dot{v} &= -mg\theta_1 - mg\theta_2 + u \\ m(\dot{v} + \ell_i\ddot{\theta}_i) &= mg\theta_i, \quad i = 1, 2 \\ y &= \theta_1 \end{aligned}$$

where v is the velocity of the cart. If we define the state variables as $x_1 = \theta_1$, $x_2 = \theta_2$, $x_3 = \dot{\theta}_1$, $x_4 = \dot{\theta}_2$, we obtain the equations $\dot{x} = Ax + Bu$, where u is the external force applied on the cart,

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_1 & -a_2 & 0 & 0 \\ -a_3 & -a_4 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{M\ell_1} \\ -\frac{1}{M\ell_2} \end{pmatrix}, \quad C = (1 \quad 0 \quad 0 \quad 0)$$

and:

$$a_1 = \frac{(M + m)g}{M\ell_1}, \quad a_2 = \frac{mg}{M\ell_1}, \quad a_3 = \frac{mg}{M\ell_2}, \quad a_4 = \frac{(M + m)g}{M\ell_2}$$