- *Quadratic forms.* Given variables x_i, $i = 1, \cdots, n$, an expression of the form $F(x) = \sum_{i,j} \phi_{ij} x_i x_j$, $\phi_{ij} \in \mathbb{R}$, is called a *quadratic form* (here x is used to denote the set of all n variables; it can also be thought of as a vector $x = [x_1, \cdots, x_n]^*$. $F(x)$ is called *positive definite* if $F(x) > 0$, for all $x \neq 0$; it is called *positive semi-definite* if $F(x) \geq 0$, for all $x \neq 0$.

  $F(x)$ can also be expressed in matrix form by means of the matrix Φ whose $(i,j)^{\text{th}}$ entry is $(\Phi)_{ij} = \frac{1}{2}(\phi_{ij} + \phi_{ji})$; notice that this matrix is $n \times n$ and symmetric: $\Phi^* = \Phi$. Thus it has an EVD of the form $\Phi = V \Lambda V^*$, where $(\Lambda)_{ii} = \lambda_i$ is its i^{th} eigenvalue, and the i^{th} column of V is the corresponding eigenvector (recall also that $VV^* = V^*V = I_n$). Thus $F(x) = x^* \Phi x$. If we define the vector $y = Vx$, then the quadratic form is diagonalized:

$$F(x) = \sum_{i=1}^{n} \lambda_i y_i^2.$$

  It readily follows that

$$\lambda_{\max} \sum_{i=1}^{n} x_i^2 \geq F \geq \lambda_{\min} \sum_{i=1}^{n} x_i^2,$$

  and furthermore, these extrema are attained if x is the eigenvector corresponding to these eigenvalues, i.e.

$$x_{\max}^* \Phi x_{\max} = \lambda_{\max} \underbrace{x_{\max}^* x_{\max}}_{\|x_{\max}\|_2^2} \quad \text{and} \quad x_{\min}^* \Phi x_{\min} = \lambda_{\min} \underbrace{x_{\min}^* x_{\min}}_{\|x_{\min}\|_2^2}.$$

  It follows that in terms of the eigenvalues of Φ, F is positive definite (semi-definite) if $\lambda_i > 0$, $i = 1, \cdots, n$ ($\lambda_i \geq 0$, $i = 1, \cdots, n$). In this case we call the matrix Φ positive definite (semi-definite), and denote this by $\Phi > 0$, ($\Phi \geq 0$), respectively.

  Notice that given a matrix $K \in \mathbb{R}^{p \times q}$, the matrices $KK^* \in \mathbb{R}^{p \times p}$ and $K^*K \in \mathbb{R}^{q \times q}$, are both symmetric and positive semi-definite.

- In the sequel we will consider dynamical systems described by state and output equations

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t),$$

  where $u(t) \in \mathbb{R}^m$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, are the *input*, *state*, *output*, respectively, and

$$A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \quad C \in \mathbb{R}^{p \times n}, \quad D \in \mathbb{R}^{p \times m}.$$

- *Reachability.* $\bar{x} \in \mathbb{R}^n$ is reachable from the zero state at time $t = \bar{T}$, if there exists an input $\bar{u}$ such that

$$\bar{x} = x(\bar{T}) = \int_0^{\bar{T}} e^{A(\bar{T}-t)} B \bar{u}(t)\, dt$$

  The set of all reachable states will be denoted by

$$X^{\text{reach}} = \{x : \text{ reachable for some } \bar{T} \text{ and } \bar{u}\}.$$

  The system is called *completely reachable* if $X^{\text{reach}} = \mathbb{R}^n$.

  We define the *reachability matrix*

$$R_n = \left[B,\ AB,\ A^2B,\ \cdots\ ,\ A^{n-1}B\right] \in \mathbb{R}^{n \times nm},$$

and the *reachability gramian*

$$P(T) = \int_0^T e^{At} BB^* e^{A^*t}\, dt \geq 0.$$

Notice that $P(T)$ is symmetric and positive semi-definite because it is the sum of symmetric and positive semi-definite matrices $\sum_t V(t)V^*(t)$, where $V(t) = e^{At}B$.

- *Reachability: Main Results.* The following holds

$$X^{\text{reach}} = \operatorname{im} R_n(A,B) = \operatorname{im} P(T), \quad T > 0$$

As a corollary, the system is completely reachable if and only if

$$X^{\text{reach}} = \mathbb{R}^n \quad \Leftrightarrow \quad \operatorname{rank} R_n(A,B) = n \quad \Leftrightarrow \quad P(T) > 0,$$

the positive definiteness condition being in turn equivalent to all eigenvalues of the reachability gramian $P(T)$ being *positive* for all $T > 0$.

Let $\bar{x} \in X^{\text{reach}}$; there exists $\bar{w}$ such that $P(\bar{T})\bar{w} = \bar{x}$. The following input

$$\bar{u}(t) = B^* e^{A^*(\bar{T}-t)} \bar{w}, \quad t \in [0, \bar{T}],$$

steers the system from the state zero at time zero, to the state $\bar{x}$ at time $\bar{T}$. Furthermore, as shown in HW#5, among all inputs which implement this transfer, $\bar{u}$ is a *minimal energy* input, i.e. its 2-norm (energy) $\|\bar{u}\|_2^2 = \int_0^{\bar{T}} u^*(t)u(t)\, dt$, is the smallest possible. In particular there holds

$$E_{\bar{x}} = \|\bar{u}\|_2^2 = \bar{w}^* P(\bar{T}) w$$

In case the system is completely reachable, the gramian is invertible and the last two formulas become

$$\bar{u}(t) = B^* e^{A^*(\bar{T}-t)} [P(\bar{T})]^{-1} \bar{x}, \quad t \in [0, \bar{T}], \quad E_{\bar{x}} = \bar{x}^* [P(\bar{T})]^{-1} \bar{x}.$$

The second formula above shows that the gramian provides a way of classifying states according to their *degree of reachability*, namely, according to the energy required to reach the corresponding state. It follows that the states which is easiest/most difficult to reach are the eigenvectors of $P(T)$ corresponding to the largest/smallest eigenvalues, respectively.

- *The infinite reachability gramian.* If A is stable, i.e. all its eigenvalues are in the LHP (left-half of the complex plane), the reachability gramian is defined for $T \to \infty$:

$$P = P(\infty) = \int_0^\infty e^{At} BB^* e^{A^*t}\, dt.$$

It turns out that this gramian satisfies the following linear matrix equation:

$$AP + PA^* + BB^* = 0,$$

which is known as a *Lyapunov equation.* Thus, if A is stable, the gramian for infinite time, can be computed simply as the solution to the above equation.

- *Observability.* The state $\bar{x}$ is called *unobservable* if with $u = 0$, $y(t) = Ce^{At}\bar{x} = 0$, for all $t > 0$. We define the *observability matrix*

$$O_n(C,A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{pn \times n},$$

and the *observability gramian*

$$Q(\bar{T}) = \int_0^{\bar{T}} e^{A^* t} C^* C e^{At} \, dt \geq 0.$$

Let X^{unobs} denote the set of all states which are unobservable. The main result states that

$$X^{\text{unobs}} = \ker O_n(C, A) = \ker Q(T), \quad T > 0.$$

The system is called *completely observable* if $X^{\text{unobs}} = 0$, which is equivalent to the following conditions:

$$X^{\text{unobs}} = 0 \quad \Leftrightarrow \quad \text{rank}\, O_n(C, A) = n \quad \Leftrightarrow \quad Q(T) > 0, \; T > 0,$$

the last condition being equivalent in turn, to all eigenvalues of the observability gramian being positive.

It is worth noting the *duality* between the concepts of reachability and observability, namely

$$[O_n(C, A)]^* = R_n(A^*, C^*).$$

This means that the transpose of the observability matrix of the pair (C, A), is the same as the reachability matrix of the pair (A^*, C^*). Thus if the roles of the *inputs* and the *outputs* of a system are interchanged, reachability goes over into observability and vive versa.

By means of the gramian we can now classify the states according to their *degree of observability*, that is the energy of the output y caused by the initial condition x:

$$\|y\|_2^2 = \int_0^T y^*(t) y(t) \, dt = x^* \underbrace{\left[\int_0^T e^{A^* t} C^* C e^{At} \, dt \right]}_{Q(T)} x = x^* Q(T) x.$$

Thus the states which are easiest/most difficult to observe at time T, are the eigenvectors of $Q(T)$ corresponding to its biggest/smallest eigenvalues, respectively.

- The infinite observability gramian. If A is stable, i.e. all its eigenvalues are in the LHP (left-half of the complex plane), the observability gramian is defined for $T \to \infty$:

  $$Q = Q(\infty) = \int_0^{\infty} e^{A^* t} C^* C e^{At} \, dt$$

  It turns out that similarly to P, Q satisfies the Lyapunov equation

  $$A^* Q + Q A + C^* C = 0.$$

  Thus, again, if A is stable, the observability gramian for infinite time can be computed simply as the solution to the above linear matrix equation.