Analysis of the issue of consistency in identification for robust control

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Abstract

Given measured data generated by a discrete-time linear system we propose a model consisting of a linear, time-invariant system affected by norm-bounded perturbation. Under mild assumptions, the plants belonging to the resulting uncertain family form a convex set. The approach depends on two key parameters: an a priori given bound of the perturbation, and the input used to generate the data. It turns out that the size of the uncertain family can be reduced by intersecting the model families obtained by making use of different inputs. Two model validation problems in this identification scheme are analyzed, namely the worst and the best invalidation problems. It turns out that while the former is a max-min optimization problem subject to a spherical constraint, the latter is a quadratic optimization problem with a quadratic and a convex constraint.

For a given energy level, the worst invalidation problem computes the family of models which can be invalidated for all possible noise sequences; i.e. the union of all possible uncertain families is obtained. The best invalidation problem, on the other hand, yields the family of those models which can never be invalidated for any possible input of fixed energy and perturbation; in other words the intersection of all uncertain families is computed. An important consequence of the best invalidation problem is that for finite length measurements *not* all models can be invalidated, using fixed-energy inputs.

Key words: Identification, robust control, convex model family, norm-bounded perturbations, linear fractional representations, model validation, model consistency, quadratic optimization.

1 Introduction

Robust control theory assumes the existence of a nominal plant together with uncertainty, often taken to be norm bounded. A number of approaches to system identification which attempt to fit this mold have been proposed in the past decade. One of these approaches belongs to the general category of *set membership identification* which aims at characterizing the membership set of the parameters consistent with the data, the model structure and the perturbation characteristics. Set membership identification including error bounded identification has been extensively studied in the literature; see e.g. [24], [19], [20], [10], [21], [5], [6], [36], [37], [9].

We propose an approach to robust identification which seeks to provide a nominal model together with a norm-bounded perturbation. More precisely, in this paper we propose an approach which can be cast into the set membership framework. It is assumed that measured input/output data is available; with some a priori knowledge, an upper bound ϵ on the norm of the perturbation which affects the data. We assume that ϵ is such that the uncertainty in the model is convex and parametric; we will call this, the *low-noise* case. With these ingredients, an uncertain family of plants is obtained; this family is *convex*, in particular, it is an ellipsoid in parameter space. Consequently, it is given in terms of a linear fraction, the parameter being a contraction. This lends itself well to the application of robust control methods. The parameter however is *real*, and the robust control problem for this family becomes tractable since it can be converted into a quasi-convex optimization problem with linear matrix inequality (LMI) constraints [26].

We formulate two different, but related, invalidation problems, namely the worst and the best invalidation problems. For a given level of energy of the input and a bound on the perturbation, while the former computes the union of all uncertain families, the latter yields the intersection of all uncertain families. Hence the worst invalidation problem gives the models which can always be invalidated, that is, for all possible input and the noise sequences; on the other hand the best invalidation problem gives the models which can never be invalidated, that is cannot be invalidated for any input and noise sequences. Using the validation result for a single model, we reformulate the worst validation problem in a new and computationally tractable way, namely as a max-min problem subject to a spherical constraint. It should be mentioned at this point that max-min or min-max problems arise naturally in problems involving the optimal estimation of parameters under bounded uncertainty; a recent paper in this vein is [8]. A necessary and sufficient condition is derived on the energy of the input required which is guaranteed to invalidate a given model for all possible noise sequences with a given upper bound on the norm of the perturbation. The set of invalidatable models shrinks as the energy of the input sequence increases. But even for infinite length measurements, not all models can be invalidated for **all** possible noise sequences

On the other hand, the best invalidation problem is formulated as a quadratic optimization problem with two convex constraints. Since there exist no explicit solution for this, we relax the problem and obtain an upper and a lower bound of the solution, that is for the intersection of all uncertain families. An important consequence of the best invalidation problem is that for finite length measurements the intersection is never empty. Hence for a fixed upper bound ϵ, using finite-energy inputs not all the models can be invalidated. Furthermore, the true model can never be invalidated with an input of finite energy. Below we summarize the main ingredients of the proposed approach.

Main ingredients

We conclude the introduction by listing the main ingredients of this approach.

1. **Data**. Starting point is a set of data (measurements)

$$\hat{u}_t,\ \hat{y}_t \in \mathbb{R},\ t = 0, 1, 2, \cdots, N$$

2. **Assumptions**.

   (a) It is assumed that the data was generated by a discrete-time, linear, time-invariant, finite-dimensional system, and the external variables are measured with noise.

   (b) The key assumption of this framework has to do with the perturbation: it is assumed that the perturbation affecting the measured variables is norm bounded;

$$\| \text{perturbation} \| < \epsilon.$$

   (c) It is also assumed that preprocessing of the data, has yielded the order n of the underlying system.

   (d) Finally, it is assumed that ϵ is *low* (see corollary 2.2). This implies that the model uncertainty is convex and parametric.

3. **Uncertain family**. It follows that the set of plants $\mathcal{F}$ satisfying these assumptions is convex and can be parameterized as in figure 3, with δ real and bounded uncertainty.

4. **Multiple sets of data**.
$$\left.\begin{array}{ccccc} u^{(1)} & \rightarrow & y^{(1)} & \Rightarrow & \mathcal{F}_1 \\ u^{(2)} & \rightarrow & y^{(2)} & \Rightarrow & \mathcal{F}_2 \end{array}\right\} \Rightarrow \mathcal{F}_{12} = \mathcal{F}_1 \cap \mathcal{F}_2$$
In other words, the uncertain set of plants corresponding to 2 sets of data is the intersection of the sets corresponding to each experiment separately. Consequently, uncertainty can be reduced by intersection.

5. **Consistency**. The question thus arises: what plants can be *invalidated*. We formulate two different, but related, invalidation problems: (1) What is the family of the models which can be invalidated for all input of given energy and all possible perturbations; i.e. *what is the union of all families $\mathcal{F}$ by letting u vary over all possible input sequences of fixed length and, possibly, fixed energy?* (2) What is the family of models which can never be invalidated for any possible input of given energy and any perturbation; in other words *what is the intersection of all families $\mathcal{F}$ obtained by letting u vary over all possible input sequences of fixed length and, possibly, fixed energy?* The main purpose of this paper is to elucidate the **issue of the consistency described above**. This problem will be analyzed assuming that the true model is a priori known. While the first problem leads to a **max-min optimization problem** which is solved in theorem 3.2, the second one yields a **quadratic optimization problem with a quadratic and a convex constraint**.

Our approach bears many resemblances with [25]. The main difference between the two papers consists in the assumption on the noise bound: in [25] it is assumed namely, that the *amplitude* of the noise is bounded, while in our case the *energy* is bounded.

The rest of the paper is organized as follows. In section 2 an overview of the proposed identification method is presented. Section 3 formulates the worst and best invalidation problems and contains the main results. It addresses the issue of model validation, and the role of ϵ. The paper concludes with four illustrative examples in section 4. In an appendix, the basic issues concerning the non-homogeneous quadratic optimization problem on the ball are summarized.

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2 The identification framework

Consider the following input-output data records:

$$\hat{u}_t,\ \hat{y}_t \in \mathbb{R},\ t = 0, 1,\ \cdots,\ N, \tag{2.1}$$

generated by a linear discrete-time dynamical system. In this paper we seek to model the data as generated by a *linear, time-invariant, discrete-time* system together with an *additive* perturbation. This implies that the data can be decomposed in the form:

$$\hat{u} = u_t + \tilde{u}_t,\ \ \hat{y} = y_t + \tilde{y}_t$$

where the data (u_t, y_t) is generated by a linear time-invariant system and $(\tilde{u}_t, \tilde{y}_t)$ is the perturbation.

Consider a model which is compatible with (i.e. may have generated) the input-output data (u_t, y_t):

$$q(\sigma)y_t + p(\sigma)u_t = 0 \tag{2.2}$$

for appropriate values of t, where σ denotes the forward shift operator, and

$$p(\xi) := p_0 + p_1\xi +\ \cdots\ + p_m\xi^m,\ \ q(\xi) := q_0 + q_1\xi +\ \cdots\ + q_n\xi^n,\ \ p_i,\ q_i \in \mathbb{R}$$

where $m \leq n$, are two specified nonnegative integers. Also, the model is usually represented by its *transfer function*

$$H(z) = -\frac{p_0 + p_1 z +\ \cdots\ + p_m z^m}{q_0 + q_1 z +\ \cdots\ + q_n z^n},$$

or its *coefficient vector*

$$x := (p_0\ p_1\ \ \cdots\ \ p_m\ q_0\ q_1\ \ \cdots\ \ q_n)^T. \tag{2.3}$$

Equivalently, (2.2) can be written as a difference equation:

$$(p_0 u_k + p_1 u_{k+1} + \cdots + p_m u_{k+m}) + (q_0 y_k + q_1 y_{k+1} + \cdots + q_n y_{k+n}) = 0 \tag{2.4}$$

for $k = 0, 1, \ldots, s-1$, where $s := N - n + 1$. In order to write these equations in matrix form, we define the *input matrix* U and *output matrix* Y

$$U = \begin{bmatrix} u_0 & u_1 & \cdots & u_{s-1} \\ u_1 & u_2 & \cdots & u_s \\ \vdots & \vdots & \ddots & \vdots \\ u_m & u_{m+1} & \cdots & u_{m+s-1} \end{bmatrix} \in \mathbb{R}^{(m+1)\times s}, \; Y = \begin{bmatrix} y_0 & y_1 & \cdots & y_{s-1} \\ y_1 & y_2 & \cdots & y_s \\ \vdots & \vdots & \ddots & \vdots \\ y_n & y_{n+1} & \cdots & y_{n+s-1} \end{bmatrix} \in \mathbb{R}^{(n+1)\times s},$$

which are Hankel matrices. The **data matrix** is:

$$\mathcal{M} := \frac{1}{\sqrt{N}} \begin{bmatrix} U \\ Y \end{bmatrix} \in \mathbb{R}^{r\times s}, \; r := m + n + 2. \tag{2.5}$$

To guarantee that the number of rows of $\mathcal{M}$ is not more than that of columns, without loss of generality we assume that $N \geq m + 2n + 1$. It readily follows from the difference equation (2.4) that the coefficient vector x in (2.3) is in the *left kernel* of the data matrix $\mathcal{M}$, i.e.,

$$x^T \mathcal{M} = 0.$$

Motivated by this observation we define the noisy data matrix $\hat{\mathcal{M}}$ in a way similar to $\mathcal{M}$. We will study the uncertain modeling problem for the case where

$$\hat{\mathcal{M}} = \mathcal{M} + \tilde{\mathcal{M}}, \tag{2.6}$$

where $\mathcal{M}$ has a non-trivial left kernel, and the perturbation is *bounded* as follows:

$$\boxed{\| \tilde{\mathcal{M}} \|_2 \; < \; \epsilon} \tag{2.7}$$

where $\| \cdot \|_2$ denotes the 2-induced norm (the largest singular value).

Remark 2.1 **(a)** In (2.6) the matrix $\tilde{\mathcal{M}}$, called the *perturbation matrix,* has the same structure as $\mathcal{M}$ and $\hat{\mathcal{M}}$. The parameter ϵ in (2.7) is thus an upper bound of the norm of $\tilde{\mathcal{M}}$.

(b) The coefficient vector x in (2.3) completely determines the model. In our formulation, all coefficients (including the highest q_n) are allowed to vary. Later we will show how to parameterize the family of uncertain models (denoted below by $\mathcal{F}$) under the normalization constraint $q_n = 1$. ■

We are now ready to describe the family of uncertain models consistent with the data records (2.1), the model structure (2.2) and the perturbation characteristic (2.7):

$$\mathcal{F} := \left\{ x \in \mathbb{R}^r, \, x \neq 0; \; \exists \, \mathcal{M}, \text{ such that } x^T \mathcal{M} = 0 \text{ and } ||\hat{\mathcal{M}} - \mathcal{M}||_2 < \epsilon \right\}$$

Assume that a decomposition as in (2.6) exists, and $x \in \mathbb{R}^r$ is an element in the left kernel of $\mathcal{M}$. Then

$$\frac{||x^T \hat{\mathcal{M}}||_2}{||x||_2} = \frac{||x^T (\mathcal{M} + \tilde{\mathcal{M}})||_2}{||x||_2} = \frac{||x^T \tilde{\mathcal{M}}||_2}{||x||_2} \; < \; \epsilon.$$

This leads to an equivalent description of the model family $\mathcal{F}$:

$$\mathcal{F} = \left\{ x \in \mathbb{R}^r; \; \mu(x, \hat{\mathcal{M}}) := \frac{||x^T \hat{\mathcal{M}}||_2}{||x||_2} < \epsilon \right\}, \tag{2.8}$$

where $\mu(x, \hat{\mathcal{M}})$ is called the *misfit function* (see [3]) between the uncertain model x and $\hat{\mathcal{M}}$.

We now give an explicit characterization of the model family $\mathcal{F}$ in terms of the singular values and the (left) singular vectors of the data matrix $\hat{\mathcal{M}}$. Let $\hat{\mathcal{M}}$ have the following singular value decomposition (SVD):

$$\hat{\mathcal{M}} = W\Sigma V^T = \begin{bmatrix} w_r & w_{r-1} & \cdots & w_2 & w_1 \end{bmatrix} \begin{bmatrix} \hat{\sigma}_r & & & 0 \\ & \ddots & & \vdots \\ & & \hat{\sigma}_1 & 0 \end{bmatrix} V^T$$

where $W \in \mathbb{R}^{r\times r}$, and $V \in \mathbb{R}^{s\times s}$ are orthogonal matrices. The characterization of the uncertain family below can be found in [3]:

Lemma 2.1 *If $\hat{\sigma}_{k+1} > \epsilon > \hat{\sigma}_k$, then $x \in \mathcal{F}$ if and only if there exist real numbers δ_i and γ_i satisfying $\sum_{i=k+1}^{r} \delta_i^2 - \sum_{i=1}^{k} \gamma_i^2 \leq 0$, such that $x = \sum_{i=1}^{k} \beta_i\gamma_i w_i + \sum_{i=k+1}^{r} \alpha_i\delta_i w_i$, where $\alpha_i = \frac{1}{\sqrt{\hat{\sigma}_i^2 - \epsilon^2}}$, $\beta_j = \frac{1}{\sqrt{\epsilon^2 - \hat{\sigma}_j^2}}$, $i = k+1, \cdots, r$, $j = 1, \cdots, k$.*

Corollary 2.1 *If $\hat{\sigma}_2 > \epsilon$, then $x \in \mathcal{F}$ if and only if there exist a constant c such that $x = cx_c$ and*

$$x_c = w_1 + \alpha_2\delta_2 w_2 + \cdots + \alpha_r\delta_r w_r \;\text{ where }\; \alpha_i = \sqrt{\frac{\epsilon^2 - \hat{\sigma}_1^2}{\hat{\sigma}_i^2 - \epsilon^2}},\; i = 2, \cdots, r$$

and $\delta_2^2 + \cdots + \delta_r^2 \leq 1$.

In the sequel we will exclusively make use of the family of uncertain plants obtained in the Corollary 2.1 above. Next, we examine this family under the normalization constraint $q_n = 1$ in (2.3). Towards this goal, let

$$\hat{\mathcal{M}} = \begin{bmatrix} \hat{\mathcal{M}}_1 \\ \hat{m}_2^T \end{bmatrix}, \quad \hat{m}_2^T : \text{ last row of} \hat{\mathcal{M}}, \tag{2.9}$$

and define

$$A := \hat{\mathcal{M}}_1\hat{\mathcal{M}}_1^T - \epsilon^2 I_{r-1},\; b := \hat{\mathcal{M}}_1\hat{m}_2,\; c := \hat{m}_2^T\hat{m}_2 - \epsilon^2$$

or equivalently,

$$\begin{bmatrix} A & b \\ b^T & c \end{bmatrix} = \hat{\mathcal{M}}\hat{\mathcal{M}}^T - \epsilon^2 I_r$$

Now, define the set:

$$\mathcal{E} := \{\theta \in \mathbb{R}^{r-1};\; \theta^T A\theta + 2b^T\theta + c < 0\} \tag{2.10}$$

The *normalized* model family $\mathcal{F}_1$ can be expressed as follows:

$$\mathcal{F}_1 := \{x \in \mathcal{F};\; x_r = 1\} = \left\{ x = \begin{bmatrix} \theta \\ 1 \end{bmatrix} \in \mathbb{R}^r;\; \theta \in \mathcal{E} \right\}$$

The set $\mathcal{E}$ is called the *uncertain parameter set.* Thus the normalized uncertain model family $\mathcal{F}_1$ is completely determined by the uncertain parameter set $\mathcal{E}$.

Next we characterize the model family $\mathcal{F}_1$, or equivalently, the uncertain parameter set $\mathcal{E}$ by studying the quadratic inequality given in (2.10). It turns out that under certain assumptions, $\mathcal{E}$ is an ellipsoid. Let $\hat{\mathcal{M}}_1$ have the singular value decomposition (SVD):

$$\hat{\mathcal{M}}_1 = T \begin{bmatrix} \sigma_{r-1} & & & 0 \\ & \ddots & & \vdots \\ & & \sigma_1 & 0 \end{bmatrix} Z^T,$$

where $T := [\tau_{r-1} \ldots \tau_1] \in \mathbb{R}^{(r-1)\times(r-1)}$, $Z \in \mathbb{R}^{s\times s}$ are orthogonal matrices (the τ_i's are left singular vectors of $\hat{\mathcal{M}}_1$) and $0 \leq \sigma_1 \leq \cdots \leq \sigma_{r-1}$.

Corollary 2.2 *If $\epsilon < \sigma_1$, then A is positive definite and $b^T A^{-1} b - c > 0$, and the set $\mathcal{E}$ is an ellipsoid with center $\theta_c = -A^{-1}b$ and half axis lengths*

$$\alpha_i = \sqrt{\frac{b^T A^{-1} b - c}{\sigma_i^2 - \epsilon^2}}, \quad i = 1, \ldots, r-1.$$

Furthermore, $x \in \mathcal{F}$ if and only if $x = [\theta^T \;\; 1]^T$, and θ has the form:

$$\theta = -A^{-1}b + \sum_{i=1}^{r-1} \alpha_i \delta_i \tau_i \text{ where } \delta_1^2 + \cdots + \delta_{r-1}^2 \leq 1$$

Proof. First we note that by construction $\epsilon > \hat{\sigma}_1(\hat{\mathcal{M}})$. Therefore the assumption $\epsilon < \sigma_1(\hat{\mathcal{M}}_1)$ yields $\hat{\sigma}_1(\hat{\mathcal{M}}) < \epsilon < \sigma_1(\hat{\mathcal{M}}_1)$. Using the Schur complement decomposition, we have

$$\hat{\mathcal{M}}\hat{\mathcal{M}}^T - \epsilon^2 I_r = \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} = \begin{bmatrix} I_{r-1} & \\ b^T A^{-1} & 1 \end{bmatrix} \begin{bmatrix} A & \\ & c - b^T A^{-1} b \end{bmatrix} \begin{bmatrix} I_{r-1} & A^{-1}b \\ & 1 \end{bmatrix}$$

According to the interlacing eigenvalues theorem for bordered matrices [14], the smallest singular value σ_1 of the matrix $\hat{\mathcal{M}}_1$ must be not bigger the second smallest singular value $\hat{\sigma}_2(\hat{\mathcal{M}})$ of $\hat{\mathcal{M}}$. Therefore, when $\hat{\sigma}_1(\hat{\mathcal{M}}) < \epsilon < \sigma_1$, we have $\hat{\sigma}_1(\hat{\mathcal{M}}) < \epsilon < \sigma_2(\hat{\mathcal{M}})$, and hence, the inertia[1] of the Hermitian matrix $\hat{\mathcal{M}}\hat{\mathcal{M}}^T - \epsilon^2\ I_r$ is $\{r-1, 0, 1\}$. It follows from the above identity and Sylvester's law of inertia that the matrices $\hat{\mathcal{M}}\hat{\mathcal{M}}^T - \epsilon^2\ I_r$ and $\operatorname{diag}[A,\ c - b^T A^{-1} b]$, have the same inertia, namely $\{r-1, 0, 1\}$. However, due to $\epsilon < \sigma_1$, the matrix $A = \hat{\mathcal{M}}_1 \hat{\mathcal{M}}_1^T - \epsilon^2\ I_{r-1}$ is positive definite, i.e., all its eigenvalues are positive, therefore, the number $c - b^T A^{-1} b$ has to be negative. This means that the set $\mathcal{E}$ is an ellipsoid [7]. ∎

Throughout the paper we assume that the uncertain model family under consideration is an ellipsoid as characterized by Corollary 2.2.

From the vectors τ_i and θ_c we can define the polynomials $n_i(z)$, $d_i(z)$, $n_c(z)$, $d_c(z)$:

$$\begin{bmatrix} -n_i(z) \\ d_i(z) \end{bmatrix} := \begin{bmatrix} 1 & z & \cdots & z^m & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & z & \cdots & z^{n-1} \end{bmatrix} \tau_i$$

$$\begin{bmatrix} -n_c(z) \\ d_c(z) - z^n \end{bmatrix} := \begin{bmatrix} 1 & z & \cdots & z^m & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & z & \cdots & z^{n-1} \end{bmatrix} \theta_c$$

which determine the uncertain models. Then the transfer function of the corresponding uncertain model has the general parameterization:

$$\frac{n_\delta(z)}{d_\delta(z)} = \frac{n_c(z) + \sum_{i=1}^{r-1} \alpha_i \delta_i n_i(z)}{d_c(z) + \sum_{i=1}^{r-1} \alpha_i \delta_i d_i(z)}, \qquad \|\delta\| \leq 1 \tag{2.11}$$

Remark 2.2 **(a)** The semi-axis lengths α_i increase with increasing ϵ (upper bound on the norm of the perturbation). For a fixed ϵ the largest semi-axis length, or the radius of the ellipsoid is α_1, is determined by the smallest singular value σ_1 of $\hat{\mathcal{M}}_1$. Moreover, the larger the smallest singular value, the smaller the radius of the ellipsoid.

(b) In the case $\epsilon \geq \sigma_{min}(\hat{\mathcal{M}}_1)$ where $\hat{\mathcal{M}}_1$ is defined in (2.9), the characterization of the uncertain family is non-convex. In this paper we will concentrate on the case $\epsilon < \sigma_{min}(\hat{\mathcal{M}}_1)$.

The *consequences* of this assumption are as follows. First implication is that the order of numerator and denominator have already been determined by preprocessing the data. Second implication is the exclusion of unmodeled dynamics. Consequently, models obtained exhibit parametric uncertainty. Finally, we would like to

[1]The *inertia* of a matrix D is a triplet of nonnegative integers $\{\nu(D), \delta(D), \pi(D)\}$ where $\nu(D)$, $\delta(D)$, and $\pi(D)$ are respectively the number of eigenvalues of D with $\mathcal{R}e\{\lambda_i(D)\} < 0$, $\mathcal{R}e\{\lambda_i(D)\} = 0$, and $\mathcal{R}e\{\lambda_i(D)\} > 0$.

stress that this assumption, does not impose any practical constraints on the signal-to-noise ratio (SNR) of the measurement variables. A simple simulation illustrates this fact, see Figure 1 below. The simple first order plant $\frac{-1}{z+1/2}$ is considered. Three different inputs are used: a sinusoidal input (whose frequency is optimal according to [2]), an impulse, and an exponential (with exponent .95). White gaussian noise is added at the output; the signal-to-noise ratio is SNR=3/2 and SNR=10. Six uncertain families result, five are ellipses, i.e. $\epsilon \geq \sigma_{min}(\hat{\mathcal{M}}_1)$, and one is a hyperbola (the exponential with SNR=3/2), because violate the assumption on ϵ. Thus the uncertain families for SNR=3/2 are much bigger than those corresponding to higher SNR. Notice also that the true plant is the point $(-1, -1/2)$ on this figure.

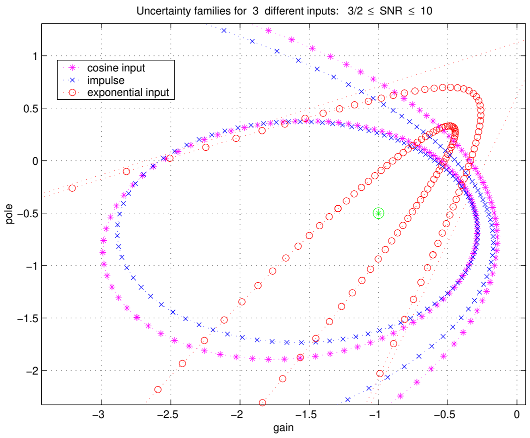

Figure 1: The uncertain families for three different inputs and two different SNR values.

(c) The transfer function (2.11) is a linear fraction in the parameters δ_i, i.e., δ_i appear linearly in the numerator and denominator.

(d) Many algorithms have been developed to characterize the set exactly or approximately $\mathcal{E}$ under the assumption that the measurements are affected by bounded noise. These methods are known as *error-bounded estimation.* For details, see e.g. [24], [19], [20], [21], [5], [6].

(e) The difference between the family $\mathcal{F}$ and $\mathcal{F}_1$ is shown in figure 2. Thus while the former is a cone, the latter is the intersection of this cone with a hyper-plane, which gives the ellipsoid $\mathcal{E}$.

(f) One can determine the uncertain model family corresponding to *multiple* sets of data. Let the sets of input-output data $\hat{w}_{ki}$, $i = 0, 1, \ldots, N_k$ (cf. (2.1)) and the error bounds ϵ_k, be available, and let ϵ_k satisfy the assumption $\epsilon < \sigma_1(\hat{\mathcal{M}}_1)$. The corresponding normalized uncertain model family $\mathcal{F}_1$ is the intersection of the uncertain model families $\mathcal{F}_1^k$:

$$\mathcal{F}_1 = \bigcap_k \mathcal{F}_1^k = \bigcap_k \left\{ x \in \mathbb{R}^r;\ \mu(x, \hat{\mathcal{M}}_k) < \epsilon_k \right\}.$$

The strategy of intersecting the model families has the advantage that it reduces the size of the uncertain model family. However, the membership set corresponding to multiple sets of data, although still convex, is no longer described in a simple way. To remedy this situation, one usually approximates the membership set by an ellipsoid in some optimal sense. In our case, the membership set is an ellipsoid for every set of data. Therefore, the membership set corresponding to multiple sets of data is in effect the intersection of several ellipsoids. Efficient algorithms have been developed [7], [30] for computing various suboptimal outer approximations of the intersection of ellipsoids, by formulating the problem as a convex optimization problem with linear matrix inequality (LMI) constraints.

(g) Robust Stability: Now, we briefly review the robust synthesis problem. The goal is to find a controller which maximizes the stability margin. The uncertain plant is given by (2.11), where δ is a *real* parameter uncertainty. From (2.11), the transfer function of every model in the family $\mathcal{F}_1$ can be expressed compactly in the form:

$$G_\delta(z) = \frac{\tilde{N}(z) + \delta^T \tilde{N}_\delta(z)}{\tilde{M}(z) + \delta^T \tilde{M}_\delta(z)} \tag{2.12}$$

where $\tilde{M}$, $\tilde{N}$, $\tilde{M}_\delta$, $\tilde{N}_\delta$ are all known stable transfer functions (the first two are scalar, and the last two are column vectors of size $r - 1$); furthermore $\tilde{M}$ and $\tilde{N}$ are assumed to be coprime, and δ is a real contractive vector of size $r - 1$, representing the uncertainty. The uncertain plants with linear fractional transfer function (2.12) can be represented as in Figure 3, where the transfer function

$$G = \frac{1}{\tilde{M}^2} \begin{bmatrix} -\tilde{M}_\delta \tilde{M} & \tilde{N}_\delta \tilde{M} - \tilde{M}_\delta \tilde{N} \\ \tilde{M}^2 & \tilde{N}\tilde{M} \end{bmatrix},$$

has size $r \times 2$. Using the ideas first developed in [1], and subsequently used in [26], all controllers that simultaneously stabilize the uncertain model family, can be obtained by solving a quasi-convex optimization problem.

■

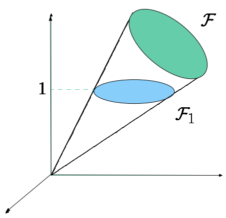

Figure 2: The families $\mathcal{F}$ and $\mathcal{F}_1$

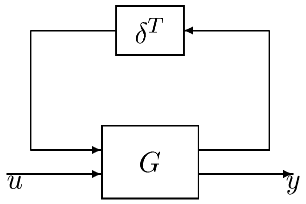

Figure 3: Block diagram of the uncertain plant with $||\delta|| \leq 1$

3 Model validation and consistency: an analysis

The problems of consistency and model validation have been studied extensively starting with Ljung's book [15]. However, since we have a different framework, these issues need to be reexamined. The main result of our identification method asserts that given a level of misfit ϵ and an input u, all models x with misfit (see (2.8)) satisfying $\mu(x, \hat{\mathcal{M}}) < \epsilon$ are *not* invalidated by the input u and corresponding output y. On the other hand, all models x satisfying $\mu(x, \hat{\mathcal{M}}) \geq \epsilon$ are indeed *invalidated*. Moreover, the true model belongs to the *intersection* of any 2 uncertain families $\mathcal{F}$ and it can never be invalidated.

In light of these remarks, the issue of *consistency* in our framework can be stated as follows: *Given a certain model described by the model parameter $\bar{x}$ (2.3), does there exist an input function u such that this model is invalidated, i.e. such that $\mu(\bar{x}, \hat{\mathcal{M}}) \geq \epsilon$?* Based on this question, we state two invalidation problems:

1. **The worst invalidation problem:** What is the family of the models which can be invalidated for all input of given energy and all possible perturbations; i.e. *what is the union of all ellipsoids by letting u vary over all possible input sequences of fixed length and, possibly, fixed energy?*

2. **The best invalidation problem:** What is the family of model which can never be invalidated for any possible input of given energy and any perturbation; in other words words *what is the intersection of the ellipsoids obtained by letting u vary over all possible input sequences of fixed length and, possibly, fixed energy?*

Below, we will address these issues.

Throughout this section, we denote the true model by x_0. Expression (2.4) shows that every $y(t)$ for $t \geq 0$ can be written as a linear combination of the initial conditions y_{-n}, y_{-n+1}, $\cdots$, y_{-1} and the values of the input sequence $u_{-n}, u_{-n+1}, \cdots, u_{t-n+m}$. In the sequel, we will *assume* for simplicity that *the system is relaxed*, i.e. all initial conditions are zero and no input is applied until the zeroth time step, $y_t = u_t = 0$ for $t < 0$; and that *the input is noise-free*, i.e. $\tilde{U} = 0$. The assumption that the system is relaxed will yield $y_0 = y_1 = \cdots = y_{n-m-1} = 0$; i.e. there will be as many zero outputs as the relative degree of the true model x_0. So, we begin the observations at the $(n-m)^{th}$ time step. Hence the data matrix $\mathcal{M}$ can be written as

$$\mathcal{M} = u_0\mathcal{M}_0 + u_1\mathcal{M}_1 + \cdots + u_{N-n+m}\mathcal{M}_{N-n+m}$$

where all $\mathcal{M}_i$ are constant matrices given in terms of x_0 (in particular $\mathcal{M}_k = \sigma^k\mathcal{M}_0$, where σ is an appropriate shift by one column). Furthermore from (2.5), (2.6) follows

$$\tilde{\mathcal{M}} = \tilde{y}_{n-m}\tilde{\mathcal{M}}_0 + \tilde{y}_{n-m+1}\tilde{\mathcal{M}}_1 + \cdots + \tilde{y}_N\tilde{\mathcal{M}}_N$$

Proposition 3.1 *Given a model described by the coefficient vector $\bar{x}$, there exist matrices $\mathcal{L}$ and $\mathcal{L}_n$ such that*

$$\bar{x}^T\mathcal{M} = \underbrace{\frac{1}{\sqrt{N}}[u_0\ \ u_1\ \ u_2\ \ \cdots\ \ u_{N-n+m}]}_{=:\mathbf{u}}\mathcal{L}(x_0,\bar{x}), \quad \bar{x}^T\tilde{\mathcal{M}} = \underbrace{\frac{1}{\sqrt{N}}[\tilde{y}_{n-m}\ \ \tilde{y}_{m-m+1}\ \ \cdots\ \ \tilde{y}_N]}_{=:\mathbf{n}}\mathcal{L}_n(\bar{x}) \tag{3.1}$$

where $\mathbf{u}$, $\mathbf{n} \in \mathbb{R}^{1\times\ell}$, $\mathcal{L}(x_0,\bar{x})$, $\mathcal{L}_n(\bar{x}) \in \mathbb{R}^{\ell\times s}$, $\ell = N-n+m+1$, and $s = N-2n+m+1$. Consequently $\mathcal{M}$, $\hat{\mathcal{M}}$, and $\tilde{\mathcal{M}}$ depend linearly on $\mathbf{u}$ and $\mathbf{n}$ as $\mathcal{M}(\mathbf{u})$, $\tilde{\mathcal{M}}(\mathbf{n})$, and $\hat{\mathcal{M}}(\mathbf{u},\mathbf{n})$.

The proof of this proposition is by direct computation and is omitted. The misfit of $\bar{x}$ can thus be expressed as:

$$\mu(\bar{x},\mathcal{M}) = \frac{\|\kappa\|}{\|\bar{x}\|} \ \text{ where } \ \kappa = \mathbf{u}\mathcal{L}(x_0,\bar{x}) + \mathbf{n}\mathcal{L}_n(\bar{x}) \tag{3.2}$$

Equality $\bar{x}^T\mathcal{M} = \mathbf{u}\mathcal{L}(x_0,\bar{x})$ and the fact that the null space of $\mathcal{M}$ has dimension one (which is a consequence of the assumption that $\epsilon < \sigma_{min}(\hat{\mathcal{M}}_1)$ imply that $\mathcal{L}(x_0,\bar{x})$ has full rank for any given $\bar{x}$ where $\bar{x} \neq x_0$; moreover

$$\mathcal{L}(x_0,x_0) = 0$$

Let $\mathcal{L}(x_0,\bar{x})$ have the following SVD:

$$\mathcal{L} = B\Pi C^T \ \text{ where } \ B \in \mathbb{R}^{\ell\times\ell},\ \ \Pi = [P^T\ \ 0]^T \in \mathbb{R}^{\ell\times s},\ \ P = \operatorname{diag}(\pi_i) \in \mathbb{R}^{\ell\times\ell},\ \ \pi_i\ > \pi_{i+1},\ \ C \in \mathbb{R}^{s\times s}. \tag{3.3}$$

In the sequel we will assume for simplicity that the singular values π_i of $\mathcal{L}(x_0,\bar{x})$ are distinct for $\bar{x} \neq x_0$.

As stated earlier, we will consider two different invalidation problems, namely the *worst* and the *best* invalidation problems formulated below in Section 3.1 and Section 3.2, respectively.

3.1 The worst invalidation problem:

In this invalidation problem, we compute all models $\bar{x}$ which can be invalidated for all possible input sequences $\mathbf{u}$ and the noise sequences $\mathbf{n}$. In other words we find the models $\bar{x}$ which are outside the union of all possible ellipsoids. We know that a given model $\bar{x}$ is invalidated by the data if, and only if, its misfit with respect to $\hat{\mathcal{M}}$ is larger than or equal to ϵ, i.e. $\mu(\bar{x},\hat{\mathcal{M}}) \geq \epsilon$. Hence we seek to compute the family

$$\boxed{\mathcal{W} := \{\bar{x} :\ \mu(\bar{x},\hat{\mathcal{M}}) \geq \epsilon\ \forall\mathbf{u},\mathbf{n} \text{ such that } \|\mathbf{u}\| \leq \tau \text{ and } \|\tilde{\mathcal{M}}\| < \epsilon\}.} \tag{3.4}$$

It follows from (3.2) that the system $\bar{x}$ is invalidated by this set of measurements if, and only if, $\|\mathbf{u}\mathcal{L}(x_0,\bar{x}) + \mathbf{n}\mathcal{L}_n(\bar{x})\| \geq \epsilon\|\bar{x}\|$. The following optimization problem results; it looks for the solutions $\mathbf{u}$ on the boundary of this set:

$$\min\|\mathbf{u}\| \ \text{ such that } \ \|\mathbf{u}\mathcal{L}(x_0,\bar{x}) + \mathbf{n}\mathcal{L}_n(\bar{x})\| = \epsilon\|\bar{x}\|,\ \ \forall\mathbf{n}. \tag{3.5}$$

If the above minimum is less than τ in (3.4), the given model $\bar{x}$ belongs to $\mathcal{W}$; otherwise it does not.

3.2 The best invalidation problem:

In the best invalidation problem, we compute the models $\bar{x}$ which cannot be invalidated for any possible input sequences $\mathbf{u}$ and the noise sequences $\mathbf{n}$; i.e. we find the models $\bar{x}$ which belong to the intersection of all possible ellipsoids. Hence we compute the family

$$\boxed{\mathcal{B} := \{\bar{x} : \ \mu(\bar{x}, \hat{\mathcal{M}}) < \epsilon \ \forall \mathbf{u}, \mathbf{n} \text{ such that } \|\mathbf{u}\| \leq \tau \text{ and } \|\tilde{\mathcal{M}}\| < \epsilon\}.} \tag{3.6}$$

In this case the following optimization problem results:

$$\max_{\mathbf{u},\,\mathbf{n}} ||\mathbf{u}\mathcal{L}(x_0, \bar{x}) + \mathbf{n}\mathcal{L}_n(\bar{x})\| \text{ such that } \|\mathbf{u}\| \leq \tau \text{ and } \|\tilde{\mathcal{M}}\| < \epsilon.$$

If the above maximum is less than $\epsilon\|\bar{x}\|$, the given model $\bar{x}$ belongs to $\mathcal{B}$. It follows from the definition of $\mathcal{W}$ and $\mathcal{B}$ that

$$\mathcal{B} \subseteq \mathcal{W}.$$

To simplify the notation, thereafter we let $\rho := \epsilon\|\bar{x}\|$ and $\rho_b := \|\mathbf{n}\mathcal{L}_n\|$.

3.3 Solution of the worst invalidation problem

We first consider the noise-free case.

3.3.1 The noise-free case

In the noise-free case, i.e. $\mathbf{n} = 0$, the minimization problem in (3.5) reduces to the problem

$$\vartheta_0^* = \min \|\mathbf{u}\| \quad \text{subject to} \quad \|\mathbf{u}\mathcal{L}\| = \rho \tag{3.7}$$

The following result holds[2].

Theorem 3.1 The noise-free case. *The solution of the minimization problem (3.7) is:*

$$\vartheta_0^* = \frac{\rho}{\pi_1} \tag{3.8}$$

and the corresponding optimal input is given by $\mathbf{u}^* = \frac{\rho}{\pi_1} b_1^T$ *where* π_1 *is the largest singular value of* $\mathcal{L}(x_0, \bar{x})$*, and* b_1 *is the corresponding left-singular vector.*

3.3.2 The general case

Let $\mathcal{S}_1$ denote the set of noise sequences $\mathbf{n}$ so that the main assumption (2.7) of our framework, namely $\|\tilde{\mathcal{M}}\| < \epsilon$ holds, i.e. $\mathcal{S}_1 = \{\mathbf{n} : \text{ such that } \|\tilde{\mathcal{M}}\| < \epsilon\}$. Notice that $\mathcal{S}_1$ is convex. We will determine a worst-case minimal **energy** which is required to invalidate a given model $\bar{x}$, for **all** possible noise sequences. We seek to solve the following worst-case problem

$$\boxed{\vartheta_1^* = \max_{\mathbf{n} \in \mathcal{S}_1} \min \|\mathbf{u}\| \quad \text{subject to} \quad \|\mathbf{u}\mathcal{L}(x_0, \bar{x}) + \mathbf{n}\mathcal{L}_n(\bar{x})\| = \rho} \tag{3.9}$$

Since $\mathbf{n}\mathcal{L}_n = \bar{x}^T\tilde{\mathcal{M}}$, it simply follows from the assumption $\| \tilde{\mathcal{M}} \|_2 < \epsilon$, that

$$\rho_b := \|\mathbf{n}\mathcal{L}_n\| < \epsilon\|\bar{x}\| =: \rho \tag{3.10}$$

[2]In the sequel $(\,\cdot\,)^*$ denotes an optimal quantity and not the transpose of $(\,\cdot\,)$.

Denote by $\mathcal{S}_2$ the set of noise sequences which satisfy $\|\mathbf{n}\mathcal{L}_n\| < \rho$, i.e. $\mathcal{S}_2 = \{\mathbf{n} : \text{ such that } \|\mathbf{n}\mathcal{L}_n\| < \rho\}$. Again, this is convex. Using the result $\|\mathbf{n}\mathcal{L}_n\| < \rho$ we formulate another worst-case optimization problem:

$$\boxed{\vartheta_2^* = \max_{\mathbf{n}\in\mathcal{S}_2} \min \|\mathbf{u}\| \quad \text{subject to} \quad \|\mathbf{u}\mathcal{L}(x_0, \bar{x}) + \mathbf{n}\mathcal{L}_n(\bar{x})\| = \rho} \tag{3.11}$$

The optimization problem (3.11) yields a worst-case minimal **energy** which is required to invalidate a given model $\bar{x}$, under **all** possible noise sequences with $\|\mathbf{n}\mathcal{L}_n\| < \rho$.

Proposition 3.2 *There holds $\mathcal{S}_1 \subseteq \mathcal{S}_2$. Consequently, The optimal energy in the first case is less than that in the second case, i.e.*

$$\vartheta_1^* \leq \vartheta_2^*.$$

The following gives the solution of the problem (3.11).

Lemma 3.1 *Let $\bar{x}$ be the given model to be invalidated and $\mathcal{L}$, $\mathcal{L}_n$ and $\mathbf{n}$ be defined by (3.1) and (3.2). Let $\mathcal{L}$ have the SVD (3.3), and $\mathbf{n}\mathcal{L}_n$ be norm bounded by ρ. Then the solution ϑ_2^* of the problem (3.11) is the same as the solution of the minimization problem (3.7). Consequently, its solution is given by (3.8), i.e.*

$$\vartheta_2^* = \frac{\rho}{\pi_1}. \tag{3.12}$$

Notice that $\mathbf{n} = 0$ satisfies the condition $\|\tilde{\mathcal{M}}\| < \epsilon$, i.e. $0 \in \mathcal{S}_1$. Moreover, for $\mathbf{n} = 0$ the original consistency problem (3.9) reduces to the noise-free problem (3.7) and its solution is given by (3.8). Thus the upper bound in Proposition 3.2 is indeed attained. Combining this with Lemma 3.1, we have just proved the main result of this section:

Theorem 3.2 **The general case**. *Let $\bar{x}$ be the given model to be invalidated and $\mathcal{L}$, $\mathcal{L}_n$ and $\mathbf{n}$ be defined by (3.1) and (3.2). Let $\mathcal{L}$ have the SVD (3.3), and $\mathbf{n}$ be such that $\|\tilde{\mathcal{M}}\| < \epsilon$. The solution ϑ_1^* of the original consistency problem (3.9) is the same as the solution of the minimization problem (3.11), consequently the same as the solution of the minimization problem (3.7). Hence, its solution is given by (3.8), i.e.*

$$\vartheta_1^* = \frac{\rho}{\pi_1}.$$

Corollary 3.1 *There is no single input $\mathbf{u}$, $\|\mathbf{u}\| \leq \frac{\rho}{\pi_1}$ which invalidates $\bar{x}$ for all possible noise sequences $\mathbf{n}$.*

Remark 3.1 $(\mathbf{a})$. Theorem 3.2 states that the worst-case minimal energy, which is required to invalidate a given model $\bar{x}$ for **all** possible noise sequences, is $\frac{\rho}{\pi_1}$. But, we note that unlike the noise-free case, the **input** depends on the noise sequence $\mathbf{n}$; see Corollary 3.1
$(\mathbf{b})$. Theorem 3.2 also states that the worst-case minimal energy ϑ_1^*, which is required to invalidate the given model for all noise sequences $\mathbf{n} \in \mathcal{S}_1$, is the same as the minimal energy ϑ_0^* for $\mathbf{n} = 0$. To be more general, the worst-case minimal energy ϑ_1^* is attained for noise sequences $\mathbf{n}$ satisfying $\mathbf{n}\mathcal{L}_n = 0$. In other words let $\mathcal{L}_n = [\,Q_1 \;\, Q_2\,][\,\Omega^T \;\, 0\,]^T Y^T$ be the SVD of $\mathcal{L}_n$. Then ϑ_1^* is attained for all noise sequences $\mathbf{n}$ with $\mathbf{n} \in Span(Q_2^T)$; see the proof of Lemma 3.1, **Step 3**. It should be noticed that unless $\mathcal{L}$ is zero, the energy ϑ_1^* is always finite.
$(\mathbf{c})$. For the worst invalidation problem, the worst possible case is if we misjudge the perturbation. Thus the effect of the perturbation is maximized, if $\mathbf{n}\mathcal{L}_n = 0$ or equivalently $\bar{x}^T\tilde{\mathcal{M}} = 0$. ■

Theorem 3.2 implies that not all models can be invalidated using inputs of fixed (finite) energy for **all** possible noise sequences $\mathbf{n}$. Furthermore, from the relationship $\mathcal{L}(x_0, x_0) = 0$, follows that the true model x_0 can never be invalidated using an input of finite energy. We note that Theorem 3.2 holds for infinite length measurements, because in this case $\mathcal{L}$ is well defined and so is its largest singular value π_1. This means that even for infinite length measurements the worst-case minimal energy, $\vartheta_1^* = \frac{\rho}{\pi_1}$, required to invalidate a given model $\bar{x}$ for **all** possible noise sequences, is nonzero; for a concrete illustration, see subsection 3.3.3 part **5.**. Similar results can be found in [18], [23], [13] and [29] in the context of robust identification. Hence as in the finite length measurements case, these observations immediately yield

Corollary 3.2 *If $\epsilon \neq 0$, the set of plants which cannot be invalidated using inputs of fixed (finite) energy, for* **all** *possible noise sequences, does not shrink to the true model as $N \to \infty$.*

3.3.3 Special Cases

1. From the above considerations follows that the model $\bar{x}$ cannot be invalidated by an input $\mathbf{u}$ for **all** noise sequences $\mathbf{n}$ *if, and only if,* $\|\mathbf{u}\| < \vartheta_1^* = \frac{\epsilon\|\bar{x}\|}{\pi_1}$. Hence the model $\bar{x}$ cannot be invalidated by an input $\mathbf{u}$ for **all** noise sequences $\mathbf{n}$ *if, and only if,* $\bar{x}$ belongs to the family $\mathcal{D} = \left\{\bar{x} : \pi_1(\mathcal{L}) < \frac{\epsilon\|\bar{x}\|}{\|\mathbf{u}\|}\right\}$. ■

2. **Model validation when $u = 0$**. If the system is excited only by initial conditions $y(0)$, $y(1)$, $\cdots$, $y(n-1)$, the noise-free data matrix has the form: $\mathcal{M} = \begin{bmatrix} 0 & Y^T \end{bmatrix}^T$, where Y is defined in (2.5). Let $\bar{x} = [\bar{p}_0 \ \cdots \ \bar{p}_m \ \underbrace{\bar{q}_0 \ \cdots \ \bar{q}_{n-1} \ 1}_{\bar{\mathbf{q}}^T}]^T$, be a given model; it follows that $\bar{x}^T\mathcal{M} = \bar{\mathbf{q}}^T Y$. Hence for a given model $\bar{x}$ whose transfer function has the same denominator as the true model, $\bar{x}^T M = 0$ regardless of the numerator. Then from (3.1), it follows that $\mathcal{L}(x_0, \bar{x}) = 0$ for this given $\bar{x}$. Therefore, this observation leads to the conclusion that a given model $\bar{x}$ whose transfer function has the same denominator as x_0 cannot be invalidated by using initial conditions alone. ■

3. **Model validation in practice**. In the analysis above we have assumed that the true model x_0 is known. Here we provide a method for invalidating a given model $\bar{x}$ when x_0 is unknown.

   Let $u(t)$ be some input sequence and $\hat{\mathcal{M}} = \mathcal{M} + \tilde{\mathcal{M}}$ be the corresponding data matrix. We need to find an input u_γ and a corresponding data matrix $\hat{\mathcal{M}}_\gamma$ so that $\|\bar{x}^T\hat{\mathcal{M}}_\gamma\| \geq \epsilon\|\bar{x}\|$. Choose the input u_γ as $u_\gamma = \gamma u$ where γ is a constant. We feed this input to the same model and the corresponding data matrix has the form $\hat{\mathcal{M}}_\gamma = \gamma\mathcal{M} + \tilde{\mathcal{M}}$. A moment's reflection shows that for large enough $\mid \gamma \mid$ the condition $\|\bar{x}^T\hat{\mathcal{M}}_\gamma\| \geq \epsilon\|\bar{x}\|$ will hold. Indeed, for every choice of γ so that $\mid \gamma \mid \geq \frac{2\,\epsilon\|\bar{x}\|}{\|\,\bar{x}^T\mathcal{M}\|}$, $\bar{x}$ is invalidated. Notice that γ is always finite unless $\bar{x} = x_0$. The size of γ will provide some information about how close the model $\bar{x}$ to be invalidated is to the true plant x_0. This observation can serve as a basis for an identification algorithm. We will not further elaborate on this.

4. **Infinite length measurements**. Finally, we examine the infinite length measurements case in more detail. The following result holds.

   Proposition 3.3 *For infinite length measurements with zero initial conditions, $\mathcal{L}\mathcal{L}^T$ is a symmetric, positive definite Toeplitz matrix.*

   The proof is by direct computation and is omitted. An interesting consequence of this is that for infinite length measurements, $\mathcal{L}$ is the convolution operator of a linear system $\mathcal{S}$ whose transfer function is given by

$$G_{\bar{x},x_0}(z) = \frac{1}{z^n}\left[\mathbf{p}(z) - G_{x_0}(z)\mathbf{q}(z)\right]. \tag{3.13}$$

where $\mathbf{p}(z)$ and $\mathbf{p}(z)$ are, respectively, the numerator and denominator of the transfer function $G_{\bar{x}}$ corresponding to the parameter vector $\bar{x}$; and $G_0(z)$ is the transfer function of the true model x_0. Hence, $\mathcal{M}^T\bar{x}$ is the output of the system $\mathcal{S}$ for the input sequence $\mathbf{u}$. In this case, the largest singular value $\pi_1(\mathcal{L})$ becomes the $\mathcal{H}_\infty$ norm of the transfer function $G_{\bar{x},x_0}(z)$.

■

5. **The first order case for infinite length measurements**. We now turn our attention to the first order case. Given is the true model with transfer function $G(z) = \frac{b}{z-a}$. Let $\bar{x} = [-\bar{b} \;\; -\bar{a} \;\; 1]^T$ be the given model to be invalidated. Denote by $\alpha = a - \bar{a}$, and $\beta = b - \bar{b}$, respectively, the deviations from the nominal pole a and the nominal gain b respectively. As discussed in the first topic above, $\bar{x}$ cannot be invalidated for **all** possible noise sequences $\mathbf{n}$ *if and only if* it belongs to the family

$$\mathcal{D} = \left\{ \bar{x} : \; \pi_1(\mathcal{L}) < \frac{\rho}{\|\mathbf{u}\|} \right\}$$

where π_1 is the maximum singular value of $\mathcal{L}$.

As stated in the previous topic above, for $N \to \infty$, $\mathcal{L}$ is the convolution operator of the linear system with transfer function given by (3.13). For the first order case, these formulas yield:

$$G_{\bar{x},x_0}(z) = \frac{-\beta z + b\bar{a} - \bar{b}a}{z - a}.$$

In this case, $\pi_1(\mathcal{L})$ becomes the $\mathcal{H}_\infty$ norm of the above transfer function. It follows that

$$\begin{array}{lclll} \|G_{\bar{x},x_0}(z)\|_\infty & = & \mid \beta + \alpha\frac{b}{1-a} \mid & \varpi = 0 & \text{if} \quad b\alpha(\beta(1-a^2) + ab\alpha)) > 0 \\ \|G_{\bar{x},x_0}(z)\|_\infty & = & \mid \beta - \alpha\frac{b}{1+a} \mid & \varpi = \pi & \text{if} \quad b\alpha(\beta(1-a^2) + ab\alpha)) < 0, \end{array}$$

where ϖ is the frequency at which $G_{\bar{x},x_0}(z)$ attains its $\mathcal{H}_\infty$ norm. Hence, the set of plants that require *at least* unit energy to be invalidated lie inside the area bounded by the curve

$$[\beta \;\; \alpha \;\; 1] \begin{bmatrix} (1-a)^2(1-\epsilon)^2 & (1-a)b & (1-a)^2\epsilon^2 b \\ (1-a)b & b^2 - (1-a)^2\epsilon^2 & (1-a)^2\epsilon^2 a \\ (1-a)^2\epsilon^2 b & (1-a)^2\epsilon^2 a & -(1-a)^2\epsilon^2(1+a^2+b^2) \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \\ 1 \end{bmatrix} = 0 \qquad (3.14)$$

in the former case, and by the curve

$$[\beta \;\; \alpha \;\; 1] \begin{bmatrix} (1+a)^2(1-\epsilon)^2 & -(1+a)b & (1+a)^2\epsilon^2 b \\ -(1+a)b & b^2 - (1+a)^2\epsilon^2 & (1+a)^2\epsilon^2 a \\ (1+a)^2\epsilon^2 b & (1+a)^2\epsilon^2 a & -(1+a)^2\epsilon^2(1+a^2+b^2) \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \\ 1 \end{bmatrix} = 0 \qquad (3.15)$$

in the latter case. A moment's reflection shows that the above curves are hyporbolas in the α-β plane. Furthermore, the former equation corresponds to Regions 1 and 3, while the latter equation corresponds to Regions 2 and 4 in figure 6. This is a concrete illustration of the union of all ellipsoids for infinite length measurements. Notice finally that the contour in figure 6 can also be obtained as the cross section of surface 4 for unit energy inputs. ■

3.4 Solution of the best invalidation problem

As stated in Section 3.2, the best invalidation problem reduces to the problem

$$\vartheta_b^* = \max_{\mathbf{u},\,\mathbf{n}} ||\mathbf{u}\mathcal{L}(x_0, \bar{x}) + \mathbf{n}\mathcal{L}_n(\bar{x})\| \;\text{ such that }\; \|\mathbf{u}\| \leq \tau \text{ and } \|\tilde{\mathcal{M}}\| < \epsilon. \tag{3.16}$$

Then the family of plants which cannot be invalidated for any $\mathbf{u}$ and $\mathbf{n}$ is obtained as

$$\mathcal{B} = \{\bar{x} : \vartheta_b^*(\bar{x}) < \rho\}.$$

Recall that the family $\mathcal{S}_1 = \{\mathbf{n} : \text{ such that } \|\tilde{\mathcal{M}}\| < \epsilon\}$ is convex in $\mathbf{n}$. Hence the maximization problem (3.16) is a quadratic optimization problem with a quadratic constraint in $\mathbf{u}$ and a convex constraint $\mathbf{n}$. Many authors have examined the optimization of a quadratic function. Without the constraint in $\mathbf{n}$, the problem is simply the maximization of a quadratic on the ball and its solution is given by [28]. But with the addition of the second constraint the problem becomes harder and has not been examined yet. In general only necessary or sufficient conditions exits [32], and the complete solutions have been obtained only for some special cases [12], [16], [17], [31], [33]. Recently Ye and Zhang [34] have solved the problem of maximizing a homogeneous quadratic function subject to homogeneous quadratic constraints by using semidefinite relaxation as outlined below. The purpose of this section is to provide an upper and a lower bound for the solution of the best invalidation problem by means of semidefinite relaxation.

3.4.1 Optimization of homogeneous quadratic functions [34]

The problem is

$$\begin{array}{ll} \text{minimize} & x^T Q_0 x \\ \text{subject to} & x^T Q_1 x \leq 1 \\ & x^T Q_2 x \leq 1. \end{array} \tag{3.17}$$

We note that

$$\begin{array}{rcl} x^T Q_i x & = & \text{trace}(Q_i\, x\, x^T) \\ & = & \text{trace}(Q_i X),\ \text{rank}(X) = 1, X \geq 0. \end{array}$$

This observation immediately yields the so-called semidefinite programming relaxation (SDP) of (3.17) by dropping the rank condition:

$$\begin{array}{ll} \text{minimize} & \text{trace}(Q_0 X) \\ \text{subject to} & \text{trace}(Q_1 X) \leq 1 \\ & \text{trace}(Q_2 X) \leq 1 \\ & X \geq 0. \end{array} \tag{3.18}$$

The solution of (3.18) can be found using the LMI tool-box or semidefinite programming solvers. Ye and Zhang [34] have proved that the SDP relaxation of (3.17) admits no gap and the minimum of (3.18) is the same as the minimum of (3.17). For more details on the solution of (3.17), we refer the reader to [34].

Since in the original problem (3.16) the convex constraint $\|\tilde{\mathcal{M}}\| < \epsilon$ is hard to handle, we relax the problem and put it into the framework of Ye and Zhang [34]. Basically, we replace the family $\mathcal{S}_1$ with an inner and outer ellipsoid, and; thus obtain a lower and upper bound for ϑ_b^*. Consequently, we also obtain an upper and lower bound for the family $\mathcal{B}$, i.e the family of plants which cannot be invalidated for any input and noise sequences.

3.4.2 A Lower bound for ϑ_b^*

To obtain a lower bound for ϑ_b^*, we determine homogeneous ellipsoid contained in $\mathcal{S}_1$. In other words, we compute an ellipsoid

$$\mathcal{E}_{in} = \{\mathbf{n} : \mathbf{n}\, D_{in}\, \mathbf{n}^T \leq \epsilon^2\} \text{ such that } \mathcal{E}_{in} \subseteq \mathcal{S}_1.$$

To compute $\mathcal{E}_{in}$, we proceed as follows. The perturbation matrix $\tilde{\mathcal{M}}$ is given by (2.5) where $\tilde{U} = 0$ and

$$\tilde{Y} = \begin{bmatrix} \tilde{y}_{n-m} & \tilde{y}_{n-m+1} & \cdots & \tilde{y}_{N-n} \\ \tilde{y}_{n-m+1} & \tilde{y}_{n-m+2} & \cdots & \tilde{y}_{N-n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{y}_{2n-m} & \tilde{y}_{2n-m+1} & \cdots & \tilde{y}_N \end{bmatrix} = \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \vdots \\ \mathbf{n}_{n+1} \end{bmatrix}, \quad \mathbf{n}_i \in \mathbb{R}^{1\times(N+m-2n)}$$

Then we obtain

$$\begin{aligned} \tilde{\mathcal{M}}^T\tilde{\mathcal{M}} = \tilde{Y}^T\tilde{Y} &= \mathbf{n}_1^T\mathbf{n}_1 + \mathbf{n}_2^T\mathbf{n}_2 + \cdots + \mathbf{n}_{n+1}^T\mathbf{n}_{n+1} \\ &= \mathbf{n}J_{in}\mathbf{n}^T \end{aligned}$$

where $J_{in} = \mathrm{diag}\,(1, 2, \cdots, n, \underbrace{n+1, \cdots, n+1}_{N-2n}, n, \cdots, 2, 1)$. Hence it simply follows that

$$\|\tilde{\mathcal{M}}\|^2 \leq \mathbf{n}\, J_{in}\mathbf{n}^T.$$

Therefore $\mathcal{E}_{in} = \{\mathbf{n} : \mathbf{n}\, J_{in}\, \mathbf{n}^T \leq \epsilon^2\} \subseteq \mathcal{S}_1$. The corresponding optimization problem is to maximize v^TQ_0v for $Q_0^T = Q_0$ subject to two homogeneous constraints:

$$\begin{aligned} \vartheta_{in}^* = \; & \max && v^TQ_0v \\ & \text{subject to} && v^TQ_1v \leq \tau^2 \\ & && v^TQ_2v \leq \epsilon^2. \end{aligned} \tag{3.19}$$

where $v^T = [\; \mathbf{u} \;\; \mathbf{n} \;]$, $Q_0 = [\mathcal{L}^T \;\; \mathcal{L}_N^T]^T[\mathcal{L}^T \;\; \mathcal{L}_N^T]$, $Q_1 = \mathrm{diag}\,(I, 0)$, and $Q_2 = \mathrm{diag}\,(0, J_{in})$. It is clear that the problem (3.19) is in the framework of [34] and ϑ_{in}^* can be computed. Since $\mathcal{E}_{in} \subseteq \mathcal{S}_1$, we have $\vartheta_{in}^* \leq \vartheta_b^*$. Define

$$\mathcal{B}_1 := \{\bar{x} : \sqrt{\vartheta_{in}^*} < \rho\}.$$

Then it simply follows that $\mathcal{B} \subseteq \mathcal{B}_1$. In other words, by solving (3.19), we compute an outer bound for the intersection of all uncertain families $\mathcal{B}$.

3.4.3 An upper bound for ϑ_b^*

To obtain an upper bound for ϑ_b^*, we proceed similarly and compute an outer homogeneous ellipsoid

$$\mathcal{E}_{out} = \{\mathbf{n} : \mathbf{n}\, J_{out}\, \mathbf{n}^T \leq \epsilon^2\} \text{ such that } \mathcal{E}_{out} \supseteq \mathcal{S}_1.$$

First notice that

$$||\mathbf{n}_i^T\mathbf{n}_i|| = \mathbf{n}_i\mathbf{n}_i^T \leq \|\tilde{\mathcal{M}}\|^2.$$

Hence,

$$\sum_{i=1}^{n+1} \mathbf{n}_i\mathbf{n}_i^T \leq (n+1)\|\tilde{\mathcal{M}}\|^2.$$

It is easy to show that with $J_{out} = \frac{1}{n+1} J_{in}$,

$$\mathbf{n} J_{out} \mathbf{n}^T \leq \|\tilde{\mathcal{M}}\|^2.$$

Hence we obtain $\mathcal{E}_{out} = \{\mathbf{n} : \mathbf{n}\, J_{out}\, \mathbf{n}^T \leq \epsilon^2\} \supseteq \mathcal{S}_1$. The following optimization problem results:

$$\begin{array}{rcll} \vartheta^*_{out} & = & \max & v^T Q_0 v \\ & & \text{subject to} & v^T Q_1 v \leq \tau^2 \\ & & & v^T \bar{Q}_2 v \leq \epsilon^2. \end{array} \tag{3.20}$$

where $v^T = [\ \mathbf{u}\ \ \mathbf{n}\ ]$, $Q_0 = [\mathcal{L}^T\ \ \mathcal{L}_N^T]^T[\mathcal{L}^T\ \ \mathcal{L}_N^T]$, $Q_1 = \mathrm{diag}\,(I, 0)$, and $\bar{Q}_2 = \mathrm{diag}\,(0, J_{out})$. Since $\mathcal{E}_{out} \supseteq \mathcal{S}_1$, we have $\vartheta^*_b \leq \vartheta^*_{out}$. Once more the problem (3.20) is in the framework of [34]. Define

$$\mathcal{B}_2 := \{\bar{x} : \sqrt{\vartheta^*_{out}} < \rho\}.$$

It readily follows that $\mathcal{B} \supseteq \mathcal{B}_2$. Hence (3.20), yields an inner bound for the intersection of all uncertain families $\mathcal{B}$.

Results are shown in Example 4.2 where we compute $\mathcal{B}_1$ and $\mathcal{B}_2$ for a first order system.

Proposition 3.4 *If $\epsilon \neq 0$, for inputs of fixed (finite) energy, for finite length measurements $N < \infty$, the family $\mathcal{B}$ includes x_0. Furthermore a neighborhood of models of x_0 is also included in $\mathcal{B}$ for inputs of given energy level.*

Proof: Recall that for the true model x_0, $\mathcal{L}(x_0, x_0) = 0$. Since $\|\mathbf{n}\mathcal{L}_n\| = \rho_b < \rho$, we obtain

$$\max_{\mathbf{u},\mathbf{n}} \|\mathbf{u}\mathcal{L}(x_0, x_0) + \mathbf{n}\mathcal{L}_n\| = \max_{\mathbf{n}} \|\mathbf{n}\mathcal{L}_n\| < \rho.$$

Hence the true model can never be invalidated. Therefore $x_0 \in \mathcal{B}$. This proves that $\mathcal{B}$ is never empty. To prove the second part, we proceed as follows: Let $\bar{x}$ be the given model, which is close enough to the true model x_0, so that $\|\mathcal{L}(x_0, \bar{x})\| = \delta_{\bar{x}}$ is a small number. It simply follows that

$$\max_{\mathbf{u},\mathbf{n}} \|\mathbf{u}\mathcal{L} + \mathbf{n}\mathcal{L}_n\| \leq \|\mathbf{u}\|\delta_{\bar{x}} + \rho_b.$$

Define $\zeta := \rho - \rho_b > 0$. Since the $\|\mathbf{u}\|$ is finite, we can always find a model $\bar{x}$ which is arbitrarily close to the true model x_0 so that $\delta_{\bar{x}} < \frac{\zeta}{\|\mathbf{u}\|}$, and consequently $\bar{x}$ cannot be invalidated, i.e. $\bar{x} \in \mathcal{B}$. This completes the proof. ■

3.5 The proofs for the worst invalidation problem

This sub-section is devoted to the proof of lemma 3.1. The proof is based on propositions 3.5 and 3.6 given below.

If $\mathbf{n}$ is known explicitly for the given system $\bar{x}$, (3.5) is a non-homogeneous quadratic optimization problem on the ball. The following result describes its solution.

Proposition 3.5 *With the singular value decomposition of $\mathcal{L}(x_0, \bar{x})$ as in (3.3), the optimization problem in (3.5) is equivalent to minimizing the norm of*

$$\Phi(w) = (w - \hat{b})^T P^{-2}(w - \hat{b}) \qquad \text{such that} \qquad \|w\| = \rho,$$

where $\hat{b} = C^T \mathcal{L}_n^T \mathbf{n}^T$, and P is defined by (3.3).

Proof. It is easy to see that the minimization problem defined in (3.5) is equivalent to problem: $\min \|\hat{\mathbf{u}}\|^2$, such that $\|A\hat{\mathbf{u}} + b\|^2 = \rho^2$, where $A = \mathcal{L}^T$, $b = \mathcal{L}_n^T \mathbf{n}^T$ and $\hat{\mathbf{u}} = \mathbf{u}^T$; thus

$$\|A\hat{\mathbf{u}} + b\|^2 = \hat{\mathbf{u}}^T A^T A \hat{\mathbf{u}} + 2b^T A \hat{\mathbf{u}} + b^T b = \hat{\mathbf{u}}^T B \Pi \Pi^T B^T \hat{\mathbf{u}} + 2b^T C \Pi^T B^T \hat{\mathbf{u}} + b^T C C^T b$$

By defining $\bar{\mathbf{u}} = B^T \hat{\mathbf{u}}$ and $\hat{b} = C^T b$, we obtain $\|A\hat{\mathbf{u}} + b\|^2 = \bar{\mathbf{u}}^T \Pi \Pi^T \bar{\mathbf{u}} + 2\hat{b}^T \Pi^T \bar{\mathbf{u}} + \hat{b}^T \hat{b}$. Notice that

$$\Pi^T \bar{\mathbf{u}} = \pi_1 \bar{\mathbf{u}}(1) + \pi_2 \bar{\mathbf{u}}(2) + \cdots + \pi_s \bar{\mathbf{u}}(s)$$

Since $\{\bar{\mathbf{u}}(s+1), \cdots, \bar{\mathbf{u}}(l)\}$ does not appear in the constraint, we simply choose $\bar{\mathbf{u}}(s+1) = \bar{\mathbf{u}}(s+2) = \cdots = \bar{\mathbf{u}}(l) = 0$. Then by defining $\Gamma := [\bar{\mathbf{u}}(1) \;\; \bar{\mathbf{u}}(2) \;\; \cdots \;\; \bar{\mathbf{u}}(s)]$, the problem becomes

$$\min \|\Gamma\|^2 \;\; \text{such that} \;\; \|P\Gamma + \hat{b}\|^2 = \rho^2$$

where $P = \text{diag}\,(\pi_i)$. Finally by defining $P\Gamma + \hat{b} = w$ we obtain the problem of minimizing

$$\Phi(w) = (w - \hat{b})^T P^{-2} (w - \hat{b}) \qquad \text{such that} \qquad \|w\| = \rho$$

which is equivalent to (3.5). ∎

Proposition 3.6 *Let $\bar{x}$ be the given model to be invalidated and $\hat{b} = C^T \mathcal{L}_n^T \mathbf{n}^T$ be as defined in Proposition 3.5. The* max-min *problem defined in (3.11) is equivalent to the problem*

$$\max_{\lambda_L, \hat{b}_i} \sum_{i=1}^{s} \frac{\lambda_L^2 \, \pi_i^2 \, \hat{b}_i^2}{(1 - \lambda_L \pi_i^2)^2} \quad \text{such that} \quad \sum_{i=1}^{s} \frac{\hat{b}_i^2}{(1 - \lambda_L \pi_i^2)^2} = \rho^2 \quad \text{and} \quad \sum_{i=1}^{s} \hat{b}_i^2 = \rho_b^2 < \rho^2 \tag{3.21}$$

where ρ and ρ_b are defined by (3.10).

Proof: *Necessity*. We note that the solution to problem $\min_w \Phi(w)$ such that $\|w\| = \epsilon \|\bar{x}\|$ is unique. This simply follows applying Theorem A.1 with $B = 2P^{-2}$, $h = 2P^{-2}\hat{b}^T$ and $f = \hat{b}P^{-2}\hat{b}^T$ noticing that B is positive definite. Using (A.8) and noticing that P is diagonal, the minimum value of $\Phi(w)$ is computed as follows:

$$\begin{aligned}
\Phi(w^{\lambda_L}) &= \|P^{-1}(w^{\lambda_L} - \hat{b})\|^2 \\
&= \|P^{-1} \left[(P^{-2} - \lambda_L I)^{-1} P^{-2} \hat{b} - \hat{b} \right] \|^2 \\
&= \|P^{-1} (P^{-2} - \lambda_L I)^{-1} \lambda_L \hat{b}\|^2 \\
&= \sum_{i=1}^{s} \frac{\lambda_L^2 \, \pi_i^2 \, \hat{b}_i^2}{(1 - \lambda_L \pi_i)^2}
\end{aligned} \tag{3.22}$$

It follows from (A.5) that λ_L and $\hat{b}$ should satisfy the condition

$$\sum_{i=1}^{s} \frac{\pi_i^{-4} \hat{b}_i^2}{(\pi_i^{-2} - \lambda_L)^2} = \rho^2 \quad \Rightarrow \quad \sum_{i=1}^{s} \frac{\hat{b}_i^2}{(1 - \lambda_L \pi_i^2)^2} = \rho^2 \tag{3.23}$$

To complete the proof of the necessity, we need to show that $\|\hat{b}\| < \rho_b$. It readily follows that

$$\|b\| = \|\mathbf{n}\mathcal{L}_n\| = \|\bar{x}^T \tilde{\mathcal{M}}\| = \rho_b < \rho. \tag{3.24}$$

Combining (3.22), (3.23) and (3.24) together with $\|\hat{b}\| = \|C^T b\| = \|b\|$ completes the proof of the necessity part.

Sufficiency. Let w be the solution to $(P^{-2} - \lambda I)w = P^{-2}\hat{b}$ and $w^T w = \rho^2$. Then it follows from (A.6), (A.7) and (A.8), and Theorem A.3 that

$$\sum_{i=1}^{s} \frac{\lambda_L^2 \, \pi_i^2 \, \hat{b}_i^2}{(1-\lambda_L \pi_i)^2} \text{ such that } \sum_{i=1}^{s} \frac{\hat{b}_i^2}{(1-\lambda_L \pi_i^2)^2} = \rho^2$$

translates to the problem

$$\min \Phi(w) = (w - \hat{b})^T P^{-2} (w - \hat{b}) \text{ such that } \|w\| = \rho.$$

Then noticing $\|\hat{b}\| = \|C^T \mathcal{L}_n^T \mathbf{n}^T\| = \|\mathcal{L}_n^T \mathbf{n}^T\| < \rho$, completes the proof of the sufficiency, and consequently the proof of the proposition. ■

Proof of theorem 3.1: It follows from Proposition 3.5 that in the noise-free case, the minimization problem (3.7) reduces to the problem

$$\min \Phi(w) = \min w^T P^{-2} w \qquad \text{such that} \qquad \|w\| = \rho.$$

It is obvious that the minimizer w^* is given by $w^* = \rho e_1$ where e_1 is the first unit vector[3] , and the minimum is $\Phi^* = \frac{\rho^2}{\pi_1^2}$. Then using the definitions of $\bar{\mathbf{u}}$, $\hat{\mathbf{u}}$ and Γ introduced in the proof of Proposition 3.5, the claim follows.

Proof of lemma 3.1: The proof is given in 3 steps. In the first step the problem is solved when $\hat{b}$ has fixed norm, namely $\|\hat{b}\| = \nu$. Then, in the second step, by a simple modification the solution is generalized to the case as in (3.21) (equivalent to the max-min problem in (3.11)) where $\hat{b}$ is norm bounded $\|\hat{b}\| = \rho_b < \rho$. In the third step corresponding worst-case noise and the worst-case minimal energy are obtained.

Step 1: Assume that $\hat{b} = C^T \mathcal{L}_n^T \mathbf{n}^T$ has fixed norm as stated above, namely $\|\hat{b}\| = \nu$. First, we note that if $\nu = 0$, the problem becomes the noise-free problem, and the solution is given by (3.1). Also it follows from (3.24) that $\nu < \rho$. Hence, for the rest of Step 1, we will assume that ν is a fixed number which is greater than 0 and less than ρ. In that case, we obtain the the following optimization problem:

$$\max_{\lambda_L, \hat{b}_i} \sum_{i=1}^{s} \frac{\lambda_L^2 \, \pi_i^2 \, \hat{b}_i^2}{(1-\lambda_L \pi_i^2)^2} \qquad \text{such that} \qquad \sum_{i=1}^{s} \frac{\hat{b}_i^2}{(1-\lambda_L \pi_i^2)^2} = \rho^2, \text{ and } \sum_{i=1}^{s} \hat{b}_i^2 = \nu^2$$

The corresponding Lagrangian is equal to

$$\mathbf{L}(\lambda_L, \hat{b}_i) = \sum_{i=1}^{s} \frac{\lambda_L^2 \, \pi_i^2 \, \hat{b}_i^2}{(1-\lambda_L \pi_i^2)^2} - \mu \left(\sum_{i=1}^{s} \frac{\hat{b}_i^2}{(1-\lambda_L \pi_i^2)^2} - \rho^2 \right) - \gamma \left(\sum_{i=1}^{s} \hat{b}_i^2 - \nu^2 \right)$$

Hence, to have a stationary point, we need to satisfy

$$\frac{\partial \mathbf{L}(\lambda_L, \hat{b}_i)}{\partial \hat{b}_i} = \frac{\partial \mathbf{L}(\lambda_L, \hat{b}_i)}{\partial \lambda_L} = \frac{\partial \mathbf{L}(\lambda_L, \hat{b}_i)}{\partial \mu} = \frac{\partial \mathbf{L}(\lambda_L, \hat{b}_i)}{\partial \gamma} = 0.$$

It follows that

$$\frac{\partial \mathbf{L}(\lambda_L, \hat{b}_i)}{\partial \hat{b}_i} = \frac{2\hat{b}_i}{(1-\lambda_L \pi_i^2)^2} \left[\lambda_L^2 \pi_i^2 - \mu - \gamma (1 - \lambda_L \pi_i^2)^2 \right] = 0, \; i = 1, \cdots n. \tag{3.25}$$

Next, we take the derivative of $\mathbf{L}(\lambda_L, \hat{b}_i)$ with respect to λ_L:

$$\frac{\partial \mathbf{L}(\lambda_L, \hat{b}_i)}{\partial \lambda_L} = \sum_{i=1}^{s} \frac{2\hat{b}_i^2 \pi_i^2}{(1-\lambda_L \pi_i^2)^3} (\lambda_L - \mu) = 0$$

[3]First unit vector e_1 is $e_1 = [\ 1\ 0\ \cdots\ 0\ ]^T$.

We know that $\lambda_L < \pi_i^{-2}$, or equivalently $(1-\lambda_L\pi_i^2) > 0$. Hence this last relationship yields

$$\mu = \lambda_L$$

Plugging $\lambda_L = \mu$ into (3.25) leads to

$$\Rightarrow \hat{b}_i \left(\lambda_L + \frac{\gamma}{1-\gamma\pi_i^2}\right)\left(\lambda_L - \frac{1}{\pi_i^2}\right) = 0$$

The above equality reveals that we can have *at most* 2 non-zero $\hat{b}_i$ values. But, since $\lambda_L < \pi_i^{-2}$, this reduces to

$$\lambda_L = -\frac{\gamma}{1-\gamma\pi_i^2} \text{ and } \lambda_L = -\frac{\gamma}{1-\gamma\pi_j^2} \text{ for some } i,\, j$$

However it is clear that these equations are consistent only if $i = j$. Hence, we can have only one nonzero $\hat{b}_i$ value. Let $\hat{b}_p$ denote this non-zero $\hat{b}_i$; we obtain

$$\hat{b}_p^2 = \nu^2,\ \ \hat{b}_i = 0 \text{ if } i \neq p,\ \ \lambda_L = \frac{\gamma}{\gamma\pi_p^2 - 1}$$

The constraint $\sum_{i=1}^{s} \frac{\hat{b}_i^2}{(1-\lambda_L\pi_i^2)^2} = \rho^2$ now reduces to

$$\frac{\hat{b}_p^2}{(1-\lambda_L\pi_p^2)^2} = \rho^2 \Rightarrow (1-\lambda_L\pi_p^2) = \frac{\nu}{\rho} \Rightarrow \lambda_L = \frac{1}{\pi_p^2}\frac{\rho-\nu}{\rho}$$

γ is then equal to $\gamma = \frac{1}{\pi_p^2}\frac{\nu-\rho}{\nu}$. Finally the corresponding maximum value is obtained as

$$\frac{\lambda_L^2\pi_p^2\hat{b}_p^2}{(1-\lambda_L\pi_p^2)^2} = \frac{1}{\pi_p^4}\left(\frac{\rho-\nu}{\rho}\right)^2 \pi_p^2\,\nu^2\,\frac{\rho^2}{\nu^2} = \frac{(\rho-\nu)^2}{\pi_p^2}$$

To maximize the last expression, we cannot simply choose $\pi_p = \pi_s$. Because, with this choice, we may not meet the condition $\lambda_L < \pi_i^{-2}$. Therefore we need we need to find the largest integer k so that

$$\frac{1}{\pi_k^2}\frac{\rho-\nu}{\rho} < \pi_1^{-2} \qquad \text{and} \qquad \frac{1}{\pi_{k+1}^2}\frac{\rho-\nu}{\rho} \geq \pi_1^{-2}.$$

In this case the maximum value is

$$\max_{\lambda_L,\hat{b}_i} \sum_{i=1}^{s} \frac{\lambda_L^2\,\pi_i^2\,\hat{b}_i^2}{(1-\lambda_L\pi_i^2)^2} = \frac{(\rho-\nu)^2}{\pi_k^2}. \tag{3.26}$$

Step 2: Now we generalize the problem to the case where $\hat{b}$ is norm bounded as in the problem (3.21) which is equivalent to the original max-min problem stated in (3.11). Let $\hat{b}$ satisfy $\|\hat{b}\| = \rho_b < \rho$ where ρ_b is defined in (3.10). Basically we maximize the maximum values obtained in (3.26) with respect to ν so that $\nu \leq \rho_b < \rho$. Since we have $\rho_b < \rho$, we solve another optimization problem, namely

$$\sup_{\nu} \frac{\rho-\nu}{\pi_k} \quad \text{such that} \begin{cases} \frac{\rho-\nu}{\rho} < \frac{\pi_k^2}{\pi_1^2},\quad \frac{\rho-\nu}{\rho} \geq \frac{\pi_{k+1}^2}{\pi_1^2}, \quad \text{and} & 0 < \nu \leq \rho_b < \rho \\ \pi_k = \pi_1 & \nu = 0 \end{cases}$$

The maximum is attained in the limit when $\nu = 0$ and $\pi_k = \pi_1$. Since $\nu = 0$, in this case we obtain $\hat{b}^* = 0$, the problem reduces to the noise-free case and the maximum is given by

$$\vartheta_2^* = \frac{\rho}{\pi_1}.$$

Step 3: In this step we obtain the worst-case noise $\mathbf{n}^*$ using $\hat{b}^*$ obtained in Step 2 and the equality $\hat{b}^* = C^T \mathcal{L}_n^T (\mathbf{n}^*)^T$. Since C is unitary, $\hat{b} = 0$ yields $\mathcal{L}_n^T (\mathbf{n}^*)^T = 0$. Let $\mathcal{L}_n = [\ Q_1 \ \ Q_2\ ][\ \Omega^T \ \ 0\ ]^T Y^T$ be the SVD of $\mathcal{L}_n$. It readily follows that $\mathcal{L}_n^T (\mathbf{n}^*)^T = 0$ holds for all noise sequences $\mathbf{n}^*$ with $\mathbf{n}^* \in Span(Q_2^T)$. We note that $\mathbf{n} = 0$ satisfies $0 \in Span(Q_2^T)$, and hence $\mathbf{n} = 0$, i.e. the noise-free case yields the worst-case minimal energy. This can also seen from the equality that $\vartheta_2^* = \vartheta_0^*$. One should notice that the condition $\mathbf{n}^* \in Span(Q_2^T)$ yields also the worst-case minimal energy for the original consistency problem (3.9); i.e. $\vartheta_1^* = \frac{\rho}{\pi_1}$ is attained for $\mathbf{n}^* \in Span(Q_2^T)$. ∎

Proof of Corollary 3.1: Since $\|\mathbf{u}\| \leq \frac{\rho}{\pi_1}$, we obtain $\|\mathbf{u}\mathcal{L}\| \leq \rho$. First assume that $\|\mathbf{u}\mathcal{L}\| < \rho$. Then choosing $\mathbf{n} = 0$ yields $\|\mathbf{u}\mathcal{L} + \mathbf{n}\mathcal{L}_n\| = \|\mathbf{u}\mathcal{L}\| < \rho$; consequently $\bar{x}$ is not invalidated. Now assume that $\|\mathbf{u}\mathcal{L}\| = \rho$. If we can find a noise sequence $\mathbf{n}$ so that $\|\mathbf{u}\mathcal{L} + \mathbf{n}\mathcal{L}_n\| < \rho$, the proof will be complete. This is always possible except for the case where $\mathbf{u}\mathcal{L}$ is perpendicular to $\mathbf{n}\mathcal{L}_n, \forall \mathbf{n}$. Recall that $\mathcal{L}$ and also $\mathcal{L}_n$ are full rank matrices. Hence $\|\mathbf{u}\mathcal{L}\| = \rho$ is the surface of an s dimensional ball and $\|\mathbf{n}\mathcal{L}_n\| < \rho$ is an s dimensional open ball; both live in a space of dimension s. The condition that $\mathbf{u}\mathcal{L}$ is perpendicular to $\mathbf{n}\mathcal{L}_n$ will occur if one of these balls is degenerate; i.e. lives in lower dimensional space. But since both $\mathcal{L}$ and $\mathcal{L}_n$ have full rank, this cannot be the case. Hence the two balls $\|\mathbf{n}\mathcal{L}_n\| < \rho$ and $\|\mathbf{u}\mathcal{L} + \mathbf{n}\mathcal{L}_n\| < \rho$ have a non-empty intersection. Therefore we can always find a noise sequence $\mathbf{n} \in \mathcal{S}_1$ so that $\|\mathbf{u}\mathcal{L} + \mathbf{n}\mathcal{L}_n\| < \rho$, and hence $\bar{x}$ is not invalidated. ∎

4 Examples

In this section we will illustrate the concepts developed above by means of four examples.

4.1 The worst invalidation problem for a first order system

We consider a discrete LTI first order dynamical system described by two parameters a and b: $G(z) = \frac{b}{z-a}$. The numerical values are assumed to be $b = 1$, $a = 0.5$, and the number of data points $N = 100$. ϵ is chosen to be 0.05. We solve the minimization problem defined in (3.9). Figure 4 shows the worst-case minimum energy ϑ_1^* on a logarithmic scale, i.e. $\log_{10}(\vartheta_2^*)$, required to invalidate a given model under all possible noise sequences represented as a deviation from the true model. On that figure α stands for the deviation from the nominal pole and β stands for the deviation from the nominal gain. Notice that as the deviation from true model becomes smaller, the minimum energy ϑ_1^* increases, and finally as $\bar{x} \to x_0$, $\vartheta_1^* \to \infty$.

To illustrate the worst-case solution is attained when $\mathbf{n}\mathcal{L}_n = 0$, we choose $\mathbf{n} = 0$ for simplicity and compare the solution of problem (3.7) with the solution of (3.5). For the latter, we assume that the noise is known and is additive white Gaussian $\tilde{y}_t \sim N(0, 0.05)$; and we solve the problem using the tools developed in [27]. To obtain a two dimensional plot for good comparison, α_i and β_i are chosen so that they lie on a line and corresponding plot is reduced to a 2 dimensional one. The results are shown in Figure 5. Figure 5 reveals that the worst-case solution is attained when $\mathbf{n}$ is zero, i.e. for the noise-free case.

Figure 6 shows the curves defined in (3.14) and (3.15) for infinite length measurements. It shows that for *any* choice of a unit energy input function, all models inside the area shown are *not* guaranteed to be invalidated for **all** input $\mathbf{u}$ and the noise $\mathbf{n}$ sequences even for $N = \infty$. On the other hand, all the models outside the area can be invalidated for all possible noise sequence and inputs of unit energy. Moreover, all the models $\bar{x}_*$ on the boundary of this area need unit energy to be invalidated.

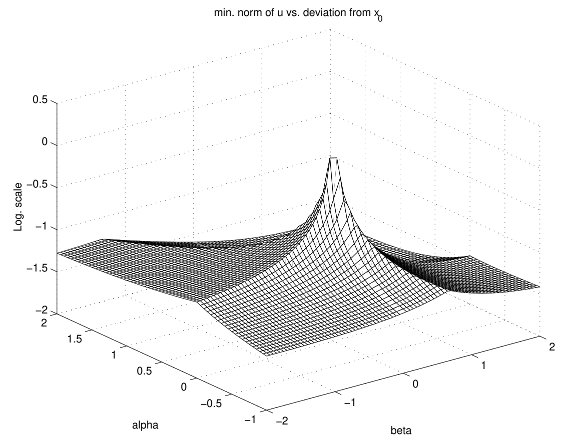

Figure 4: Minimal energy ϑ_1^* (log scale) necessary to invalidate given model

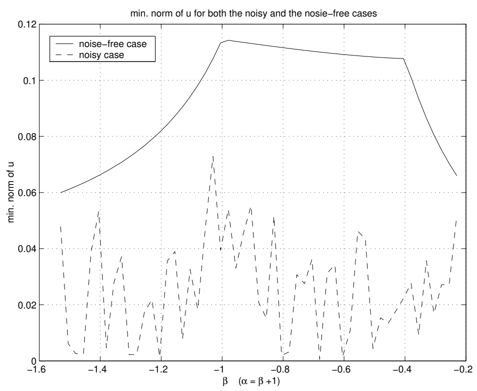

Figure 5: Comparison of the energies of the worst-case (the noise-free) and the noisy solutions

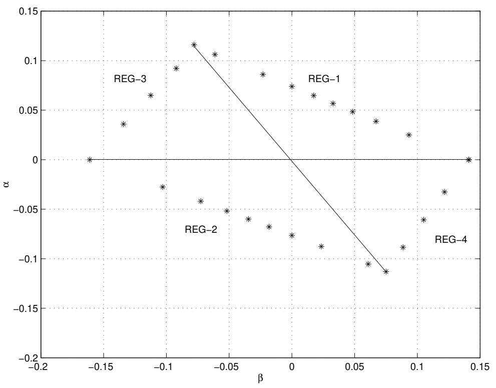

Figure 6: Curves (3.14) and (3.15) for $\vartheta_1^* = 1$: the set of parameter values which cannot be invalidated.

4.2 The best invalidation problem for a first order system

We consider a first order discrete LTI dynamical system given by its transfer function $G(z) = \frac{4}{z-0.9}$. Due to high numerical complexity of the LMI's involved, we choose $N = 15$. ϵ is taken to be 0.05. We solve the maximization problems (3.19) and (3.20); and consequently obtain an upper bound $\mathcal{B}_2$ and a lower bound $\mathcal{B}_1$ for $\mathcal{B}$, the family of model which cannot be invalidated for *any* input $\mathbf{u}$ and noise $\mathbf{n}$. We consider the unit energy inputs, i.e. $\tau = 1$ in (3.19) and (3.20). Moreover, the family $\mathcal{W}$[4] is also computed. The results are shown in Figure 7 and 8. The family $\mathcal{B}$ lies between the families $\mathcal{B}_1$ and $\mathcal{B}_2$. The Figure 7 reveals that none of the families in $\mathcal{B}_2$ can be invalidated for any noise sequences and any input of unit energy. On the other hand in Figure 8, all the families outside $\mathcal{W}$ can be invalidated for any input of unit energy and any noise sequence. The family $\mathcal{B}$ is a subfamily of $\mathcal{W}$ as expected.

4.3 The worst invalidation problem for a switching system

Given is a dynamical system which behaves by switching between two discrete LTI first order systems namely

$$G_1(z) = \frac{b_1}{z - a_1} \quad \text{and} \quad G_2(z) = \frac{b_2}{z - a_2}$$

[4]Recall that as N approaches infinity, the set $\mathcal{W}$ is bounded by hyporbolas. See Section 3.3.3 Item 5. and Figure 6 above.

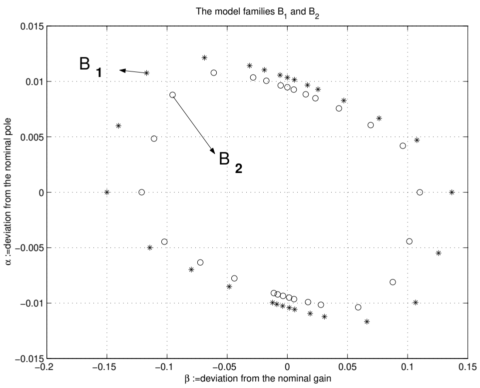

Figure 7: The families $\mathcal{B}_1$, $\mathcal{B}_2$.

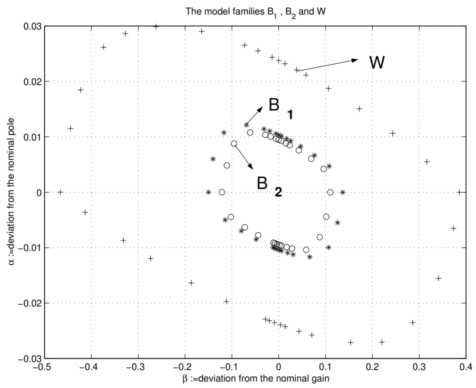

Figure 8: The families $\mathcal{W}$, $\mathcal{B}_1$, $\mathcal{B}_2$.

Thus we assume that the system behaves as $G_1(z)$ for the time interval $[0, 1, \cdots, N-1]$ and as $G_2(z)$ for the time interval $[N, N+1, \cdots, 2N]$. Moreover, we will assume that

$$a_1 - a_2 \neq 0 \quad \text{and/or} \quad b_1 - b_2 \neq 0.$$

Consequently, there is no LTI model x_0, which satisfies $x_0^T \mathcal{M} = 0$. It follows that $\mathcal{L}$ never becomes zero for any given model, and the solution to (3.9) is always finite in this case. The minimal energy of an input which will invalidate any model is finite, and hence we can define $E_{max} = \max_{\bar{x}} \vartheta_1^*(\bar{x})$, which is finite. Then, there exists a finite energy input satisfying $\|\mathbf{u}\| > E_{max}$, which invalidates *every* model for *all* possible $\mathbf{n}$. To illustrate this theoretical consideration, we choose $b_1 = 1$, $a_1 = 0.5$, $b_2 = 1.2$, $a_1 = -0.2$, and $N = 50$. We solve the minimization problem defined in (3.9). Figure 9 shows the minimum energy ϑ_1^* required to invalidate a given system represented in terms of its pole and gain. In figure 9, $\bar{a}$ stands for the pole of the given system $\bar{x}$ and $\bar{b}$ stands for its gain. Figure 9 illustrates that E_{max} occurs for $\bar{a} = 0.04$ and $\bar{b} = 1.32$ and unlike the previous example, it is *finite* for all models: $E_{max} = 1.96$. Thus by choosing an appropriate input sequence $\mathbf{u}$ such that $\|\mathbf{u}\| > E_{max}$, we can invalidate *all* candidate models $\bar{x}$.

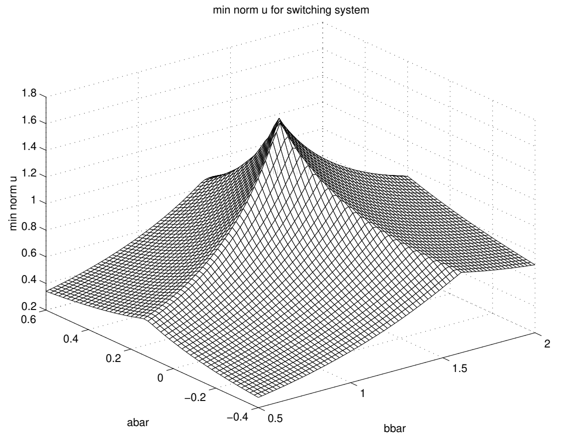

Figure 9: Minimal energy ϑ_1^* (log scale) necessary to invalidate given model

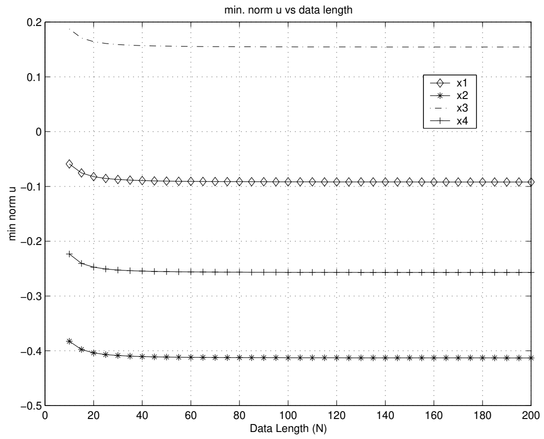

Figure 10: The worst-case minimal energy required to invalidate $\bar{x}_i$, $i = 1, 2, 3, 4$, vs data length N (section 4.4).

4.4 Model validation and the data length

In this example we examine the worst-case minimum energy ϑ_1^* as a function of the length of the measurements. Let $\bar{x}_i$ $i = 1, \cdots, 4$ be chosen as follows:

$$\bar{x}_1 = [-1.4 \ -0.9 \ 1]^T, \ \bar{x}_2 = [-0.4 \ -0.1 \ 1]^T, \ \bar{x}_3 = [-1.2 \ -0.7 \ 1]^T, \ \bar{x}_4 = [-0.95 \ 0.1 \ 1]^T$$

We solve the minimization problem (3.9) for every $\bar{x}_i$ as N varies from $N = 10, \cdots, 200$. Let $\vartheta_1^*(\bar{x}_i, N)$ be the solution for a given N for $i = 1, \cdots, 4$. Figure 10 shows $\vartheta_1^*(\bar{x}_i, N)$ corresponding to the chosen model $\bar{x}_i$ as N changes. Figure 10 indicates that even as N goes to infinity, $\vartheta_1^*(\bar{x}_i, N)$ does not go to zero; it stays constant at a finite nonzero value. This result demonstrates that even for *infinite* length measurements, a finite amount of energy is needed to invalidate a given model $\bar{x}$, and the set of plants, which cannot be invalidated for **all** noise sequences, does not shrink to the true model.

5 Conclusions

In this paper we propose a method to model sets of measured data by means of a linear, time-invariant system together with a norm-bounded perturbation. The role played by the bound of the perturbation and the input (excitation) function are analyzed. An important conclusion is that intersection of model families obtained for different input functions, provide a straightforward way of reducing the uncertainty. The issue of model validation in this framework is addressed in detail. Two problems are formulated, namely the worst and the best invalidation problems.

It follows that for finite length measurements there is a family of plants centered around the true plant, whose members cannot be invalidated for any noise sequences using inputs of fixed energy. The set of models which can be invalidated using some measurements depends on ϵ and the energy of the input u. For fixed ϵ, this set of invalidatable models decreases as the energy of either the input sequence or the initial conditions decreases (although the above considerations assume a relaxed system). Even if infinite length measurements are taken, the set of plants which cannot be invalidated for **all** possible noise sequences does not shrink to the true model.

Finally, several examples are provided to illustrate the results.

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A Appendix: Minimization of a quadratic function subject to a spherical constraint

For completeness we state here some standard results from [28] and [11].

Theorem A.1 [28] *Let $\phi(w) = f + h^T w + \frac{1}{2} w^T B w$ with $B = B^T \in \mathbb{R}^{n \times n}$; $w, h \in \mathbb{R}^n$ and $f \in \mathbb{R}$. Let $\lambda \in \mathbb{R}$, $p \in \mathbb{R}^n$ and h satisfy $(B + \lambda I)p = -h$ with $B + \lambda I$ positive semidefinite. Let $\rho \in \mathbb{R}$.*

1. *If $\lambda = 0$ and $\|p\| = \rho$, then p solves $\phi(p) = \min\{\phi(w) : \|w\| = \rho\}$.*
2. *If $\|p\| = \rho$ then p solves $\phi(p) = \min\{\phi(w) : \|w\| \leq \rho\}$.*
3. *If $\lambda \geq 0$, $\|p\| = \rho$ then p solves $\phi(p) = \min\{\phi(w) : \|w\| = \rho\}$.*

If $B + \lambda I$ is positive definite then p is unique in each of the cases (1), (2) and (3).

Theorem A.2 [11] *Let $T \in \mathbb{R}^{n \times n}$ be a symmetric square matrix and $c \in \mathbb{R}^n$ a known vector. Define $\Phi(w)$ as*

$$\Phi(w) = (w - c)^T T (w - c) \qquad \text{such that} \qquad \|w\| = \rho. \tag{A.1}$$

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the real eigenvalues of T with the corresponding eigenvectors $\{v_1, v_2, \cdots, v_n\}$ such that $T v_i = \lambda v_i$, $i = 1, \cdots, n$. Let the given c be written as

$$c = \sum_{i=1}^{n} c_i v_i \tag{A.2}$$

Then, if any real λ and vector w satisfies

$$\begin{aligned} (T - \lambda I) w &= T c && \text{(A.3)} \\ w^T w &= \rho^2, && \text{(A.4)} \end{aligned}$$

then w renders $\Phi(w)$ stationary with $\|w\| = \rho$. Let $w = \sum_{i=1}^{n} w_i v_i$. Then, (A.3) is equivalent to $\sum_{i=1}^{n} (\lambda_i - \lambda) w_i v_i = \sum_{i=1}^{n} \lambda_i c_i v_i$.

The *spectrum* of the pair (T, c), is the set of all real λ for which there exists an w such that (A.3) and (A.4) are satisfied; i.e., such that $\Phi(w)$ is stationary with $\|w\| = \rho$. The following theorem gives the complete sets of stationary values for the quadratic expression defined in (A.1).

Theorem A.3 [11] *Let $T \in \mathbb{R}^{n\times n}$ be a symmetric square matrix and $c \in \mathbb{R}^n$ a known vector; define c_i as in (A.2). Then the spectrum of (T, c) consists of all real λ such that*

$$g(\lambda) = \sum_{i=1}^{n} \frac{\lambda_i^2 \mid c_i \mid^2}{\mid \lambda_i - \lambda \mid^2} = \rho^2 \qquad (0/0 = 0,\ 1/0 = \infty), \tag{A.5}$$

together with each eigenvalue λ_k of T for which $g(\lambda_k) < \rho^2$. For each λ in the spectrum with $g(\lambda) = \rho^2$, a unique w^λ is found by solving (A.3) and (A.4). For each λ in the spectrum with $g(\lambda) < \rho^2$, there exists an $(m-1)$-sphere of w^λ with norm ρ satisfying (A.3) and (A.4), where m is multiplicity λ as an eigenvalue of T. Let the spectrum of (T, c) consists of the numbers $\{\Lambda_1, \cdots, \Lambda_p\}$ with

$$\lambda_L = \Lambda_1 < \Lambda_2 < \cdots < \Lambda_p = \Lambda_R \tag{A.6}$$

For $i = 1, 2, \cdots, p$, let w^i be any w^{Λ_i}. Then

$$\Phi(w^1) < \Phi(w^2) < \cdots < \Phi(w^p). \tag{A.7}$$

In particular, $\Phi(w^1)$, $\Phi(w^p)$ are, respectively, the minimum and maximum values of $\Phi(w)$ with $\|w\| = \rho$.

When the solution is unique, the minimizer $w^1 = w^{\lambda_L}$ for the problem $\min \Phi(w)$ such that $\|w\| = \rho$ is given by

$$w^1 = w^{\lambda_L} = (T - \lambda_L I)^{-1} T\, c. \tag{A.8}$$