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FREQUENCY DOMAIN REPRESENTATION AND SINGULAR VALUE DECOMPOSITION *

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Abstract

This contribution reviews the external and the internal representations of linear time-invariant systems. This is done both in the time and the frequency domains. The realization problem is then discussed. Given the importance of norms in control design and model reduction, the final part of this contribution is dedicated to the definition and computation of various norms. Again, the interplay between time and frequency norms is emphasized.

Key words: linear systems, internal representation, external representation, Laplace transform, $\mathcal{Z}$-transform, vector norms, matrix norms, Singular Value Decomposition, convolution operator, Hankel operator, reachability and observability gramians.

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Glossary

Notation	Meaning	First appearing
$\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$	Integers, real or complex numbers	section 2.2.1
$\mathbb{K}$	either $\mathbb{R}$ or $\mathbb{C}$	section 2.2.1
$(\,\cdot\,)^*$	Transposition and complex conjugation of a matrix	section 2.2.1
$\|\cdot\|_p$	Hölder p-norms	section 2.2.1
$\|\cdot\|_{p,q}$	matrix or operator induced norm	section 2.2.1
$\|\cdot\|_F$	Frobenius norm of matrix or operator	section 2.2.1
σ_i	Singular values	section 2.2.2
ℓ_p^n	infinite sequences of vectors in $\mathbb{K}^n$, finite p-norm	section 2.2.3
$\mathcal{L}_p^n$	functions with values in $\mathbb{K}^n$, finite p-norm	section 2.2.3
$\mathcal{D}$	(open) unit disc	section 2.2.5
h_p	vectors/matrices, analytic in $\mathcal{D}$, finite p-norm	section 2.2.5
$\mathcal{H}_p$	vectors/matrices, analytic in $\mathbb{C}_-$, finite p-norm	section 2.2.5
ℓ_∞	vectors/matrices, no poles on the unit circle, finite ∞-norm	section 2.2.5
$\mathcal{L}_\infty$	vectors/matrices, no poles on imaginary axis, finite ∞-norm	section 2.2.5
$\mathcal{L}$	Laplace transform	section 2.1.1
$\mathcal{Z}$	discrete-Laplace or $\mathcal{Z}$-transform	section 2.1.2
$\mathbb{I}$	unit step function discrete or continuous	section 2.1.2
u, x, y	Input, State, Output	section 3
h	Impulse response	section 3
H	Transfer function	section 3
$\mathbf{\Sigma}$	Linear, time-invariant system	section 3
$h * u$	convolution discrete or continuous	section 3
$\mathbb{R}[\xi]$	polynomials in ξ with real coefficients	section 3
$\mathbb{R}^{p\times q}[\xi]$	$p \times q$ matrix with entries in $\mathbb{R}[\xi]$	section 3
S_k	Markov parameters	section 3.1
$\cong$	Isomorphism	section 3.2
$\delta(t)$	Kronecker delta if $t \in \mathbb{Z}$	section 3.2.1
$\delta(t)$	delta distribution if $t \in \mathbb{R}$	section 3.2.1
$\phi(u; x_0; t)$	state at time t, initial condition x_0, with input u	section 3.2.1
$\mathcal{R}_n, \mathcal{O}_n$	Reachability, Observability matrices	section 3.2.3
$\mathcal{P}$, $\mathcal{Q}$	Reachability, Observability gramians	section 3.2.3
$\mathcal{S}$	Convolution operator	section 4.1
$\mathcal{H}$	Hankel operator	section 4.1
$\mathcal{S}^*$	Adjoint of convolution operator	section 4.2
$\mathcal{H}^*$	Adjoint of Hankel operator	section 4.3
$\|\mathbf{\Sigma}\|_{2,\infty} = \|\mathcal{S}\|_{2,\infty}$	$2, \infty$ induced norm of $\mathbf{\Sigma}$	section 4.4
$\|\mathbf{\Sigma}\|_{2,2} = \|\mathcal{S}\|_{2,2}$	$2, 2$ induced norm of $\mathbf{\Sigma}$	section 4.4
$\|\mathbf{\Sigma}\|_{\infty,\infty} = \|\mathcal{S}\|_{\infty,\infty}$	∞, ∞ induced norm of $\mathbf{\Sigma}$	section 4.4
$\|\mathbf{\Sigma}\|_H$	Hankel norm of $\mathbf{\Sigma}$	section 4.4
$\|\mathbf{\Sigma}\|_{\mathcal{H}_2} = \|H\|_{\mathcal{H}_2}$	$\mathcal{H}_2$-norm of $\mathbf{\Sigma}$	section 4.4
$\|\mathbf{\Sigma}\|_{\mathcal{L}_2} = \|h\|_{\mathcal{L}_2}$	$\mathcal{L}_2$-norm of $\mathbf{\Sigma}$	section 4.4

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1 Introduction

One of the most powerful tools in the analysis and synthesis of linear time-invariant systems is the equivalence between the *time domain* and the *frequency domain*. Thus additional insight into problems in this area is obtained by viewing them both in *time* and in *frequency*. This dual nature accounts for the presence and great success of linear systems both in engineering theory and applications.

In this contribution we will provide an overview of certain results concerning the analysis of linear dynamical systems. Time and frequency domain frameworks are inextricably connected. Therefore together with frequency domain considerations in the sequel, unavoidably, a good deal of time domain considerations are included as well.

Our goals are as follows. First, basic system representations will be introduced, both in time and in frequency. Then the ensuing *realization problem* is formulated and solved. Roughly speaking the realization problem entails the construction of a state space model from frequency response data.

The second goal is to introduce various *norms* for linear systems. This is of great importance both in *control design* and in *system approximation/model reduction*.

First it is shown that besides the convolution operator we need to attach a second operator to every linear system, namely the Hankel operator. The main attribute of this operator is that it has a discrete set of singular values, known as the *Hankel singular values*. These singular values are main ingredients of numerous computations involving control design and model reduction of linear systems. Besides the Hankel norm, we discuss various p-norms, where $p = 1, 2, \infty$. It turns out that norms which are obtained for $p = 2$ have both a time domain and a frequency domain interpretation. The rest have an interpretation in the time domain only.

The contribution is organized as follows. The next section is dedicated to a collection of useful results on two topics: The Laplace and discrete-Laplace transforms on the one hand and norms and the SVD on the other. Two tables 1, 2, summarize the salient properties of these two transforms. Section 3 develops the external and internal representations of linear systems. This is done both in the time and frequency domains, with the results summarized in two further tables 3, 4. This discussion is followed by the formulation and solution of the realization problem. The final section 4 is dedicated to the exposition of various norms for linear systems. The basic features of these norms are summarized in the fifth and last table 5. We conclude by outlining the use of the various norms in control system design and system approximation.

2 Preliminaries

2.1 The Laplace transform and the $\mathcal{Z}$-transform

The logarithm can be considered as an elementary transform. It assigns a real number to any positive real number. It was invented in the middle ages and its purpose was to convert the *multiplication* of multi-digit numbers to addition. In the case of linear, time-invariant systems the operation which one wishes to simplify is the *derivative with respect to time* in the continuous-time case or the *shift* in the discrete-time case. As a consequence, one also wishes to simplify the operation of *convolution*, both in discrete- and continuous-time. Thus an operation is sought which will transform derivation into simple multiplication in the transform domain. In order to achieve this however, the transform needs to operate on functions of time. The resulting function will be one of complex frequency. This establishes two equivalent ways of dealing with linear, time-invariant systems, namely in the *time domain* and in the *frequency domain*. In the next two section we will briefly review some basic properties of this transform, which is called *Laplace transform* in continuous-time and *discrete-Laplace* or *$\mathcal{Z}$-transform* in discrete-time.

2.1.1 Some properties of the Laplace transform

Consider a function of time $f(t)$. The *unilateral Laplace transform* of f is a function denoted by $F(s)$ of the complex variable $s = \sigma + j\omega$. The definition of F is as follows:

$$f(t) \xrightarrow{\mathcal{L}} F(s) := \int_{0^-}^{\infty} f(t)e^{-st}dt \tag{2.1}$$

Therefore the values of f for negative time are ignored by this transform. Instead, in order to capture the influence of the past, initial conditions at time zero are required (see *Differentiation in time*). The salient properties of the Laplace transform are summarized in Table 1.

2.1.2 Some properties of the $\mathcal{Z}$-transform

Consider a function of time $f(t)$, where time is discrete $t \in \mathbb{Z}$. The *unilateral $\mathcal{Z}$-transform* of f is a function denoted by $F(z)$ of the complex variable $z = re^{j\theta}$. The definition of F is as follows:

$$f(t) \xrightarrow{\mathcal{Z}} F(z) := \sum_{t=0}^{\infty} z^{-t} f(t) \tag{2.2}$$

The main features of this transform are summarized in Table 2.

2.2 Norms of vectors, matrices and the SVD

In this section we will first review some material from linear algebra which pertains to norms of vectors, norms of operators (matrices), both in finite and infinite dimensions. The latter are of importance because a linear system can be viewed as a map between infinite dimansional spaces. The Singular Value Decomposition (SVD) will also be introduced and its properties briefly discussed.

2.2.1 Norms of finite-dimensional vectors and matrices

Let X be a linear space over the field $\mathbb{K}$ which is either the field of reals $\mathbb{R}$ or that of complex numbers $\mathbb{C}$. A norm on X is a function $\nu : X \rightarrow \mathbb{R}$, such that the following three properties are satisfied. Non-strict positivity: $\nu(x) \geq 0$, $\forall\, x \in X$, with equality if $x = 0$; triangle inequality: $\nu(x+y) \leq \nu(x) + \nu(y)$, $\forall\, x, y \in X$; positive homogeneity: $\nu(\alpha x) = |\alpha|\nu(x)$, $\forall\, \alpha \in \mathbb{K}$, $\forall\, x \in X$. For vectors $x \in \mathbb{R}^n$ or $x \in \mathbb{C}^n$ the **Hölder** or p-**norms** are defined as follows:

$$\| x \|_p := \begin{cases} \left(\sum_{i \in \underline{n}} \mid x_i \mid^p\right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \max_{i \in \underline{n}} \mid x_i \mid, & p = \infty \end{cases}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \tag{2.3}$$

where $\underline{n} := \{1, 2, \cdots, n\}$, $n \in \mathbb{N}$. The 2-norm satisfies the *Cauchy-Schwartz inequality*:

$$|x^* y| \leq \| x \|_2 \| y \|_2$$

with equality holding if and only if $y = cx$, $c \in \mathbb{K}$, or $y = 0$. An important property of the 2-norm is that it is invariant under *unitary* (orthogonal) transformations; let U be such a transormation, that is, $UU^* = U^*U = I_n$. It follows that $\| Ux \|_2^2 = x^*U^*Ux = x^*x = \| x \|_2^2$. The following relationship between the Hölder norms for $p = 1, 2, \infty$ holds:

$$\| x \|_\infty \;\leq\; \| x \|_2 \;\leq\; \| x \|_1$$

One type of matrix norms are those which are *induced* by the vector p-norms defined above. More precisely

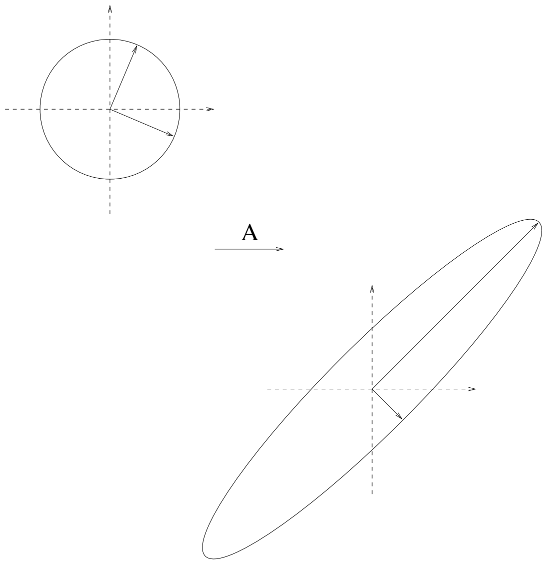


Figure 1: The linear transformation A maps the unit sphere into an ellipsoid. The singular values are the lengths of the semi-axes of the ellipsoid.

for $A \in \mathbb{C}^{m \times n}$

$$\| A \|_{p,q-\mathrm{ind}} \;:=\; \sup_{x \neq 0} \frac{\| Ax \|_q}{\| x \|_p} \tag{2.4}$$

is the *induced* p, q-*norm* of A. In particular, for $p = q = 1, 2, \infty$ the following expressions hold

$$\| A \|_1 = \max_{i \in \underline{n}} \sum_{j \in \underline{m}} |A_{ij}|, \quad \| A \|_\infty = \max_{j \in \underline{m}} \sum_{i \in \underline{n}} |A_{ij}|, \quad \| A \|_2 = [\lambda_{max}(AA^*)]^{\frac{1}{2}}$$

Besides the induced matrix norms, there exist other norms. One such class is the *Schatten p-norms* of matrices. These non-induced norms are unitarily invariant. Let $\sigma_i(A)$, $1 \leq i \leq \min(m, n)$, be the singular values of A, i.e. the square roots of the eigenvalues of AA^* (see also section 2.2.2). Then

$$\| A \|_p \;:=\; \left(\sum_{i \in \underline{m}} \sigma_i^p(A) \right)^{\frac{1}{p}}, \;\; 1 \leq p < \infty \tag{2.5}$$

It follows that the Schatten norm for $p = \infty$ is

$$\| A \|_\infty = \sigma_{max}(A)$$

which is the same as the 2-induced norm of A. For $p = 1$ we obtain the *trace norm*

$$\| A \|_1 = \sum_{i \in \underline{m}} \sigma_i(A)$$

For $p = 2$ the resulting norm is also known as the **Frobenius norm**, the **Schatten 2-norm**, or the **Hilbert-Schmidt norm** of A:

$$\| A \|_{\mathrm{F}} = \left(\sum_{i \in \underline{m}} \sigma_i^2(A) \right)^{\frac{1}{2}} = (\mathrm{trace}\,(A^*A))^{\frac{1}{2}} = (\mathrm{trace}\,(AA^*))^{\frac{1}{2}} \tag{2.6}$$

where $\mathrm{trace}\,(\cdot)$ denotes the *trace* of a matrix.

2.2.2 The singular value decomposition

Given a matrix $A \in \mathbb{K}^{n \times m}$, $n \leq m$, let the nonnegative numbers $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ be the positive square roots of the eigenvalues of AA^*. There exist unitary matrices $U \in \mathbb{K}^{n \times n}$, $UU^* = I_n$, and $V \in \mathbb{K}^{m \times m}$, $VV^* = I_m$, such that

$$A = U \Sigma V^* \ \text{ where } \ \Sigma = (\bar{\Sigma} \ \ 0) \in \mathbb{R}^{n \times m} \ \text{ and } \ \bar{\Sigma} := \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{pmatrix} \in \mathbb{R}^{n \times n} \tag{2.7}$$

The decomposition (2.7) is called the **singular value decomposition (SVD)** of the matrix A; the σ_i's are called the **singular values** of A while the columns of U, V

$$U = (u_1 \ \ u_2 \ \ \cdots \ \ u_n), \ \ V = (v_1 \ \ v_2 \ \ \cdots \ \ v_m)$$

are called the **left**, **right singular vectors** of A, respectively. These singular vectors are the *eigenvectors* of AA^*, A^*A respectively. Thus

$$Av_i = \sigma_i u_i, \ i = 1, \ \cdots, n$$

Example 2.1 Consider the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & \sqrt{2} \end{pmatrix}$. The eigenvalue decomposition of the matrices $AA^* = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}$ and $A^*A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$ are: $AA^* = U\Sigma^2 U^*$, $A^*A = V\Sigma^2 V^*$, where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ \Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \ V = \begin{pmatrix} \frac{1}{\sqrt{2}\sigma_1} & \frac{1}{\sqrt{2}\sigma_2} \\ \frac{1+\sqrt{2}}{\sqrt{2}\sigma_1} & \frac{1-\sqrt{2}}{\sqrt{2}\sigma_2} \end{pmatrix},$$

where $\sigma_1 = \sqrt{2 + \sqrt{2}}$, $\sigma_2 = \sqrt{2 - \sqrt{2}}$. Notice that A maps the unit disc in the plane to the ellipse with half-axes σ_1 and σ_2; more precisely $v_1 \mapsto \sigma_1 u_1$ and $v_2 \mapsto \sigma_2 u_2$ (see figure 1). It follows that $X = \sigma_2 u_2 v_2^*$ is a perturbation of smallest 2-norm (equal to σ_2) such that $A - X$ is singular:

$$X = \frac{1}{2} \begin{pmatrix} 1 & 1 - \sqrt{2} \\ -1 & \sqrt{2} - 1 \end{pmatrix} \ \Rightarrow \ A - X = \frac{1}{2} \begin{pmatrix} 1 & 1 + \sqrt{2} \\ 1 & 1 + \sqrt{2} \end{pmatrix}$$

■

The singular values of A are *unique*. The left-, right-singular vectors corresponding to singular values of multiplicity one are also uniquely determined (up to a sign). Thus the SVD is *unique* in case the matrix A is *square* and the singular values have *multiplicity one*.

Lemma 2.1 *The 2-induced norm of A is equal to its largest singular value $\sigma_1 = \| A \|_{2-\text{ind}}$.*

Proof. By definition

$$\| A \|_2^2 = \sup_{x \neq 0} \frac{\| Ax \|_2^2}{\| x \|_2^2} = \sup_{x \neq 0} \frac{x^* A^* A x}{x^* x}$$

Let y be defined as $y := V^* x$ where V is the matrix containing the eigenvectors of A^*A, i.e. $A^*A = V\Sigma V^*$. Substituting in the above expression we obtain

$$\sigma_n^2 \ \leq \ \frac{x^* A^* A x}{x^* x} = \frac{\sigma_1^2 y_1^2 + \ \cdots \ + \sigma_n^2 y_n^2}{y_1^2 + \ \cdots \ + y_n^2} \ \leq \ \sigma_1^2$$

This expression is maximized and equals σ_1^2, for $y = e_1$, i.e. $x = v_1$, where v_1 is the first column of V. ■

Theorem 2.1 *Every matrix A with entries in $\mathbb{K}$ has a singular value decomposition.*

Proof. Let σ_1 be the 2-norm of A; there exist unity length vectors $x_1 \in \mathbb{K}^m$, $x_1^* x_1 = 1$, and $y_1 \in \mathbb{K}^n$, $y_1^* y_1 = 1$, such that $Ax_1 = \sigma_1 y_1$. Define the unitary matrices V_1, U_1 so that their first column is x_1, y_1 respectively:

$$V_1 := [x_1 \ \ \bar{V}_1], \ \ U_1 := [y_1 \ \ \bar{U}_1]$$

It follows that

$$U_1^* A V_1 = \begin{pmatrix} \sigma_1 & w^* \\ & \\ 0 & B \end{pmatrix} =: A_1 \ \text{ where } \ w \in \mathbb{K}^{m-1}$$

and consequently

$$U_1^* A A^* U_1 = A_1 A_1^* = \begin{pmatrix} \sigma_1^2 + w^* w & w^* B^* \\ & \\ Bw & BB^* \end{pmatrix}$$

Since the 2-norm of every matrix is bigger than or equal to the norm of any of its submatrices, we conclude that $\sigma_1^2 + w^* w \leq \| AA^* \| = \sigma_1^2$. The implication is that w must be the zero vector $w = 0$. Thus

$$U_1^* A V_1 = \begin{pmatrix} \sigma_1 & 0 \\ & \\ 0 & B \end{pmatrix}$$

The procedure is now repeated for B which has size $(n-1) \times (m-1)$. ■

Assume that in (2.7) $\sigma_r > 0$ while $\sigma_{r+1} = 0$; the matrices U, Σ, V are partitioned in two blocks the first having r columns:

$$U = [U_1 \ \ U_2], \ \ \bar{\Sigma} = \begin{pmatrix} \bar{\Sigma}_1 & 0 \\ 0 & \bar{\Sigma}_2 \end{pmatrix} \ \text{ and } \ V = [V_1 \ \ V_2] \tag{2.8}$$

$$\bar{\Sigma}_1 = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix} > 0, \ \ \bar{\Sigma}_2 = 0 \in \mathbb{R}^{(n-r)\times(n-r)}$$

Corollary 2.1 *Given (2.7) and (2.8) the following statements hold.*

- $\operatorname{rank} A = r$; $\quad \operatorname{span}\operatorname{col} A = \operatorname{span}\operatorname{col} U_1$; $\quad \ker A = \operatorname{span}\operatorname{col} V_2$.
- *The orthogonal projection onto the span of the columns of A is $U_1U_1^*$. The orthogonal projection onto the kernel of A is $V_2V_2^*$.*
- *The orthogonal projection onto the orthogonal complement of the span of the columns of A is $U_2U_2^*$. The orthogonal projection onto the orthogonal complement of the kernel of A is $V_1V_1^*$.*
- **Dyadic decomposition**. *A has a decomposition as a sum of r matrices of rank one:*
$$A = \sigma_1 u_1 v_1^* + \sigma_2 u_2 v_2^* + \cdots \sigma_r u_r v_r^* \tag{2.9}$$
- *The Frobenius norm of A is* $\parallel A \parallel_{\mathrm{F}} = \sqrt{\sigma_1^2 + \cdots + \sigma_n^2} = \sqrt{\text{trace } AA^*} = \sqrt{\text{trace } A^*A}$.

For symmetric matrices the SVD can be readily obtained from the EVD (Eigenvalue Decomposition). Let the latter be: $A = V\Lambda V^*$. Define by $J := \operatorname{diag}(\operatorname{sgn}\lambda_1, \ \cdots \ , \operatorname{sgn}\lambda_n)$, where $\operatorname{sgn}\lambda$ is the *signum* function; it equals $+1$ if $\lambda > 0$, -1 if $\lambda < 0$ and 0 if $\lambda = 0$. Then $A = U\Sigma V^*$ where $U := VJ$ and $\Sigma := \operatorname{diag}(|\lambda_1|, \ \cdots \ , |\lambda_n|)$. We conclude this section with some additional properties of the SVD.

Proposition 2.1 . *Consider $A \in \mathbb{K}^{p\times q}$, $B \in \mathbb{K}^{r\times s}$. The following relationships hold true.*

1. *The singular values of A and of its complex conjugate transpose are the same: $\sigma_i(A) = \sigma_i(A^*)$*
2. *If A is square $p = q$, and invertible, then $\sigma_i(A^{-1}) = [\sigma_{p-i+1}(A)]^{-1}$, $i = 1, \cdots, p$.*
3. *If $q = r$ the smallest/largest singular values of the product AB satisfy: $\sigma_{\min}(A)\,\sigma_{\min}(B) \leq \sigma_{\min}(AB)$, and $\sigma_{\min}(A)\sigma_{\max}(B) \leq \sigma_{\max}(AB) \leq \sigma_{\max}(A)\sigma_{\max}(B)$.*
4. *If $p = r$ and $q = s$, the smallest/largest singular values of the sum $A + B$ satisfy the inequalities:*
$$\begin{aligned} &\sigma_{\max}(A) - \sigma_{\max}(B) \leq \sigma_{\max}(A+B) \leq \sigma_{\max}(A) + \sigma_{\max}(B), \\ &\sigma_{\min}(A) - \sigma_{\max}(B) \leq \sigma_{\min}(A+B) \leq \sigma_{\min}(A) + \sigma_{\max}(B). \end{aligned}$$
5. *Let $p = r$, $q = s$ and let A_{ij} denote the $(i,j)^{\mathrm{th}}$ element of A.*
$$\begin{aligned} &\max\left(\sigma_{\max}(A), \ \sigma_{\max}(B)\right) \leq \sigma_{\max}([A \ \ B]) \leq \sqrt{2}\,\max\left(\sigma_{\max}(A), \ \sigma_{\max}(B)\right), \\ &\max_{i,j} |A_{ij}| \leq \sigma_{\max}(A) \leq n \ \max_{i,j} |A_{ij}|. \end{aligned}$$

2.2.3 Norms of functions of time

In this section we will define the p-norms of infinite sequences and functions of one real variable, which in the context of system theory is taken to be *time*. The ensuing spaces are therefore *time-domain* spaces and the normsi, time-domain ones. Let $f : \mathcal{I} \longrightarrow \mathbb{K}^n$, where $\mathcal{I} \subset \mathbb{Z}$, be a sequence of vectors in $\mathbb{K}^n$ (recall that $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$). Frequent choices of $\mathcal{I}$ are: $\mathcal{I} = \mathbb{Z}$, $\mathcal{I} = \mathbb{Z}_+$ or $\mathcal{I} = \mathbb{Z}_-$. The p-norms of the elements of this space are defined as:
$$\parallel f \parallel_p := \begin{cases} \left(\sum_{t\in\mathcal{I}} \parallel f(t) \parallel_p^p\right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \sup_{t\in\mathcal{I}} \parallel f(t) \parallel_p, & p = \infty \end{cases} \tag{2.10}$$
The 2-norm of a time signal is sometimes referred to as its *energy*. The corresponding *ℓ_p spaces* are:
$$\ell_p^n(\mathcal{I}) \ := \ \{f : \ \parallel f \parallel_p < \infty\}, \ 1 \leq p \leq \infty$$

Consider now functions of a continuous variable $f : \mathcal{I} \longrightarrow \mathbb{K}^n$, where $\mathcal{I} \subset \mathbb{R}$. Frequent choices of $\mathcal{I}$ are: $\mathcal{I} = \mathbb{R}, \mathcal{I} = \mathbb{R}_+$ or $\mathcal{I} = \mathbb{R}_-$. The p-norms of f are:

$$\| f \|_p := \begin{cases} \left(\int_{t\in\mathcal{I}} \| f(t) \|_p^p \, dt\right)^{\frac{1}{p}}, \ 1 \leq p < \infty \\ \sup_{t\in\mathcal{I}} \| f(t) \|_p, \ \ p = \infty \end{cases} \tag{2.11}$$

As in the discrete-time case the 2-norm of a time signal is sometimes known as its *energy*. The corresponding $\mathcal{L}_p$ spaces are:

$$\mathcal{L}_p^n(\mathcal{I}) := \{f : \ \| f \|_p < \infty\}, \ 1 \leq p \leq \infty.$$

The spaces ℓ_p^n and $\mathcal{L}_p^n$ are often referred to as *Lebesgue spaces*.

Given a vector-valued function f of the real variable t, the definition of a norm, involves in general the definition of two separate norms, namely a *spatial norm* and a *temporal norm*. More precisely, for a fixed time instance $\hat{t}$, the spatial norm amounts to the choice of some vector norm for $f(\hat{t})$, say, $\|f(\hat{t})\|_r = \phi(\hat{t})$. The *temporal norm* then, is a norm for the scalar function $\phi(t)$, for $t \in \mathcal{I}$: $\|\phi(t)\|_s = \chi$. In this case we use the notation

$$\|f\|_{r,s} = \left(\int_{t\in\mathcal{I}} \| f(t) \|_r^s \, dt\right)^{\frac{1}{s}} = \chi$$

The corresponding *generalized* Lebesgue spaces are

$$\mathcal{L}_{r,s}^n := \{f : \ \| f \|_{r,s} < \infty\}$$

In the sequel for simplicity $\|f\|_p$ will be used when both the spatial and the temporal norms are the same, i.e., $\|f\|_p = \|f\|_{p,p}$.

2.2.4 Induced operator norms

The (p, q) norms of vector-valued functions of time defined above, *induce operator* norms on any map between Lebesgue spaces. Let for example, $T : \mathcal{L}_{p,q}^n \longrightarrow \mathcal{L}_{r,s}^m$. The (p, q)-(r, s) induced norm of T is defined in a similar way as (2.4), namely, as the largest *gain* of T given the above norms in the domain and codomain:

$$\| T \|_{(p,q)-(r,s)-\mathrm{ind}} := \sup_{u\neq 0} \frac{\|Tx\|_{r,s}}{\|x\|_{p,q}} \tag{2.12}$$

Recall that p is the spatial norm in the domain, r is the spatial norm in the codomain, q is the temporal norm in the domain, and s the temporal norm in the codomain.

2.2.5 Norms of functions of complex frequency

In this section we consider norms of functions of one complex variable. In the system theoretic context, this variable is taken to be complex frequency and the ensuing spaces and norms are *frequency-domain* ones.

Let $\mathcal{D} \subset \mathbb{C}$ denote the (open) unit disc, and let $F : \mathbb{C} \longrightarrow \mathbb{C}^{q\times r}$ be a matrix-valued function, analytic in $\mathcal{D}$. Its p-norm is defined as follows:

$$\| F \|_{h_p} := \left(\frac{1}{2\pi} \sup_{|r|<1} \int_0^{2\pi} \| F(re^{j\theta}) \|_p^p \, d\theta\right)^{\frac{1}{p}}, \ 1 \leq p < \infty$$

$$\| F \|_{h_\infty} := \sup_{z\in\mathcal{D}} \| F(z) \|_p, \ \ p = \infty$$

We will choose $\| F(z_0) \|_p$ to be the Schatten p-norm of F evaluated at $z = z_0$; however, there are other possible choices. The resulting spaces are called *Hardy h_p spaces*:

$$h_p^{q\times r} := h_p^{q\times r}(\mathcal{D}) := \{F \text{ as above with : } \| F \|_{h_p} < \infty\}$$

The following special cases are worth noting:

$$\| F \|_{h_2} = \left(\frac{1}{2\pi} \sup_{|r|<1} \int_0^{2\pi} \text{trace} \left[F^*(re^{-j\theta}) F(re^{j\theta}) \right] d\theta \right)^{\frac{1}{2}}, \quad \| F \|_{h_\infty} = \sup_{z\in\mathcal{D}} \sigma_{max} \left(F(z) \right) \tag{2.13}$$

where $\text{trace}\,(\cdot)$ denotes the *trace*, and $(\cdot)^*$ denotes complex conjugation and transposition. Let $\mathbb{C}_- \subset \mathbb{C}$ denote the (open) left half of the complex plane: $s = x + jy \in \mathbb{C}$, $x < 0$. Consider the $q \times r$ complex-valued functions F as defined above, which are analytic in $\mathbb{C}_-$. Then

$$\| F \|_{\mathcal{H}_p} := \left(\sup_{x<0} \int_{-\infty}^{\infty} \| F(x + jy) \|_p^p \, dy \right)^{\frac{1}{p}}, \ \ 1 \le p < \infty$$

$$\| F \|_{\mathcal{H}_\infty} := \sup_{z\in\mathbb{C}_-} \| F(z) \|_p, \ \ p = \infty$$

Again $\| F(s_0) \|_p$ is chosen to be the Schatten p-norm of F evaluated at $s = s_0$. The resulting spaces are the *Hardy $\mathcal{H}_p$ spaces*:

$$\mathcal{H}_p^{q\times r} := \mathcal{H}_p^{q\times r}(\mathbb{C}_-) := \{F \text{ as above with : } \| F \|_{\mathcal{H}_p} < \infty\}$$

As before, the following special cases are worth noting:

$$\| F \|_{\mathcal{H}_2} = \left(\sup_{x<0} \int_{-\infty}^{\infty} \text{trace} \left[F^*(x - jy) F(x + jy) \right] dy \right)^{\frac{1}{2}}, \quad \| F \|_{\mathcal{H}_\infty} = \sup_{s\in\mathbb{C}_-} \sigma_{max} \left(F(s) \right) \tag{2.14}$$

where, as before, $\text{trace}\,(\cdot)$ denotes the *trace*, and $(\cdot)^*$ denotes complex conjugation and transposition; furthermore

The suprema in the formulae above can be computed by means of the **maximum modulus theorem**, which states that a function f continuous inside a domain $D \subset \mathbb{C}$ as well as on its boundary ∂D and analytic inside D, attains its maximum on the *boundary* ∂D of D. Thus (2.13) and (2.14) become:

$$\| F \|_{h_2} = \left(\frac{1}{2\pi} \int_0^{2\pi} \text{trace} \left[F^*(e^{-j\theta}) F(e^{j\theta}) \right] d\theta \right)^{\frac{1}{2}} \tag{2.15}$$

$$\| F \|_{h_\infty} = \sup_{\theta\in[0,2\pi]} \sigma_{max} \left(F(e^{j\theta}) \right) \tag{2.16}$$

$$\| F \|_{\mathcal{H}_2} = \left(\int_{-\infty}^{\infty} \text{trace} \left[F^*(-jy) F(jy) \right] dy \right)^{\frac{1}{2}} \tag{2.17}$$

$$\| F \|_{\mathcal{H}_\infty} = \sup_{y\in\mathbb{R}} \sigma_{max} \left(F(jy) \right) \tag{2.18}$$

If F has no poles on the unit circle or the $j\omega$-axis, respectively, but is not necessarily analytic in the corresponding domains, the h_∞/$\mathcal{H}_\infty$ norm is not defined. Instead the ℓ_∞/$\mathcal{L}_\infty$ norm of F is defined, respectively, as follows:

$$\| F \|_{\ell_\infty} := \sup_{\theta} \sigma_{max}(F(e^{j\theta})), \quad \| F \|_{\mathcal{L}_\infty} := \sup_{y} \sigma_{max}(F(jy))$$

where in the first expression the supremum is taken over $\theta \in [0, 2\pi]$, while in the second the supremum is taken over $y \in (-\infty, \infty)$.

We conclude with the definition of two more frequency domain spaces, namely $\mathcal{L}_2[0,\ 2\pi]$ and $\mathcal{L}_2(j\omega)$. The elements of the former space space are all functions of a complex variable which are square intergrable on the unit circle, while those of the latter are square integrable on the imaginary axis:

$$\begin{aligned}\mathcal{L}_2[0,\ 2\pi] &= \{F : \mathbb{C} \longrightarrow \mathbb{C}^{p\times m},\ \text{such that (2.15)}\ < \infty\}\\ \mathcal{L}_2(j\mathbb{R}) &= \{F : \mathbb{C} \longrightarrow \mathbb{C}^{p\times m},\ \text{such that (2.17)}\ < \infty\}\end{aligned}$$

2.2.6 Connection between time and frequency domain spaces

The spaces $\ell_2(\mathcal{I})$ and $\mathcal{L}_2(\mathcal{I})$ are Hilbert spaces, that is linear spaces where not only a norm but an inner product is defined as well.[1] For $\mathcal{I} = \mathbb{Z}$ and $\mathcal{I} = \mathbb{R}$ respectively, the inner product is defined as follows:

$$\langle x, y\rangle_{\ell_2} := \sum_{t\in\mathcal{I}} x^*(t)y(t),\ \ \langle x, y\rangle_{\mathcal{L}_2} := \frac{1}{2\pi}\int_{\mathcal{I}} x^*(t)y(t)dt$$

where as before $(\cdot)^*$ denotes complex conjugation and transposition. For $\mathcal{I} = \mathbb{Z}/\mathcal{I} = \mathbb{R}$ respectively, elements (vectors or matrices) with entries in $\ell_2(\mathbb{Z})$ and $\mathcal{L}_2(\mathbb{R})$ have a *Fourier transform* defined as follows: $F(e^{j\theta}) = \sum_{-\infty}^{\infty} f(t)e^{-j\theta t}/F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$. Therefore with $z = re^{j\theta}/s = \sigma + j\omega$, $F(z)/F(s)$ belongs to frequency domain $\mathcal{L}_2$ spaces, i.e. $\mathcal{L}_2[0, 2\pi]/\mathcal{L}_2(j\mathbb{R})$, respectively. The following bijective correspondences hold:

$$\begin{aligned}\ell_2(\mathbb{Z}) = \ell_2(\mathbb{Z}_-) \oplus \ell_2(\mathbb{Z}_+) &\ \ \stackrel{\mathcal{Z}}{\longrightarrow}\ \ \mathcal{L}_2[0, 2\pi] = h_2(\mathcal{D}) \oplus h_2(\bar{\mathcal{D}})\\ \mathcal{L}_2(\mathbb{R}) = \mathcal{L}_2(\mathbb{R}_-) \oplus \mathcal{L}_2(\mathbb{R}_+) &\ \ \stackrel{\mathcal{L}}{\longrightarrow}\ \ \mathcal{L}_2(j\mathbb{R}) = \mathcal{H}_2(\mathbb{C}_-) \oplus \mathcal{H}_2(\mathbb{C}_+)\end{aligned}$$

For simplicity the above relationships are shown for spaces containing scalars. It is however equally valid for the corresponding spaces containing matrices of arbitrary dimension.

There are two results connecting the spaces introduced above. We will only state the continuous-time versions. The first has the names of *Parseval*, *Plancherel* and *Paley-Wiener* attached to it.

Proposition 2.2 *The Fourier transform $\mathcal{F}$ is a Hilbert space isometric isomorphism between $\mathcal{L}_2(\mathbb{R})$ and $\mathcal{L}_2(j\mathbb{R})$. It maps $\mathcal{L}_2(\mathbb{R}_+)$, $\mathcal{L}_2(\mathbb{R}_-)$ onto $\mathcal{H}_2(\mathbb{C}_+)$, $\mathcal{H}_2(\mathbb{C}_-)$ respectively.*

The second one shows that the $\mathcal{L}_\infty$ and $\mathcal{H}_\infty$ norms can be viewed as *induced norms*. Recall (2.4).

Proposition 2.3 **(I)** *Let $F \in \mathcal{L}_\infty$.*

(Ia) *$F\mathcal{L}_2(j\mathbb{R}) \subset \mathcal{L}_2(j\mathbb{R})$; consequently $F : \mathcal{L}_2(j\mathbb{R}) \longrightarrow \mathcal{L}_2(j\mathbb{R})$.* **(Ib)** *The $\mathcal{L}_\infty$ norm can be viewed as an induced norm in the frequency domain space $\mathcal{L}_2(j\mathbb{R})$:*

$$\| F \|_{\mathcal{L}_\infty} = \| F \|_{\mathcal{L}_2(j\mathbb{R})-\mathrm{ind}} = \sup_{X\neq 0} \frac{\| FX \|_{\mathcal{L}_2(j\mathbb{R})}}{\| X \|_{\mathcal{L}_2(j\mathbb{R})}}$$

In this last expression, $X \in \mathcal{L}_2(j\mathbb{R})$ can be restricted to lie in $\mathcal{H}_2(\mathbb{C}_+)$.

(II) *Let $F \in \mathcal{H}_\infty$.*

(IIa) *$F\mathcal{H}_2(\mathbb{C}_+) \subset \mathcal{H}_2(\mathbb{C}_+)$; thus $F : \mathcal{H}_2(\mathbb{C}_+) \longrightarrow \mathcal{H}_2(\mathbb{C}_+)$.* **(IIb)** *The $\mathcal{H}_\infty$ norm can be viewed as an induced norm both in the frequency domain space $\mathcal{H}_2(\mathbb{C}_+)$ as well as in the time domain space $\mathcal{L}_2(\mathbb{R}_+)$:*

$$\| F \|_{\mathcal{H}_\infty} = \| F \|_{\mathcal{H}_2(\mathbb{C}_+)-\mathrm{ind}} = \sup_{X\neq 0} \frac{\| FX \|_{\mathcal{H}_2(\mathbb{C}_+)}}{\| X \|_{\mathcal{H}_2(\mathbb{C}_+)}} = \sup_{x\neq 0} \frac{\| f * x \|_{\mathcal{L}_2(\mathbb{R}_+)}}{\| x \|_{\mathcal{L}_2(\mathbb{R}_+)}} = \| f \|_{\mathcal{L}_2(\mathbb{R}_+)-\mathrm{ind}}$$

[1] The spaces $\ell_p(\mathcal{I})$ and $\mathcal{L}_p(\mathcal{I})$, $p \neq 2$, do not share this property; they are Banach spaces.

3 External and internal representations of linear systems

In this section we will review some basic results concerning linear dynamical systems. Here we will assume that the external variables have been partitioned into *input variables* u and into output variables y, and will be concerned with *convolution systems*, i.e. systems where the relation between u and y is given by a convolution sum or integral

$$\mathbf{\Sigma}: \quad y = h * u \tag{3.1}$$

where h is an appropriate *weighting pattern*. This will be called the *external representation*. We will also be concerned with systems where besides the input and output variables, the *state* x has been declared as well. Furthermore, the relationship between x and u is given by means of a set of first order difference or differential equations with constant coefficients, while that of y with x and u is given by a set of linear algebraic equations. It will also be assumed that x lives in a finite-dimensional space:

$$\mathbf{\Sigma}: \quad \sigma x = Ax + Bu, \; y = Cx + Du \tag{3.2}$$

where σ is the derivative or shift operator and A, B, C, D are linear constant maps. This will be called *internal representation*.

An alternative external representation is in terms of two polynomial matrices $Q \in \mathbb{R}^{p\times p}[\xi]$, $P \in \mathbb{R}^{p\times m}[\xi]$:

$$\mathbf{\Sigma}: \quad Q(\sigma)y = P(\sigma)u \tag{3.3}$$

where as above, σ is the derivative or the backwards shift operator. It is usually assumed that $\det Q \neq 0$. This representation is given in terms of differential or difference equations linking the input and the output.

The first subsection is devoted to the discussion of systems governed by (3.1), (3.3) while the following subsection investigates some structural properties of systems represented by (3.2). These equations are solved both in the *time* and the *frequency* domains. The third subsection discusses the equivalence of the external and the internal representation, As it turns out going from the latter to the former involves the elimination of x and is thus straightforward. The converse however is far from trivial as it involves the *construction of state*. It is called the *realization problem*. This problem can be interpreted as deriving a time domain representation from frequency domain data.

3.1 External representation

A *discrete-time linear system* $\mathbf{\Sigma}$, with m input and p output channels can be viewed as a linera operator $\mathcal{S}$ which maps vector-valued sequences of dimension m to vector-valued sequences of dimension p. Thus there exists a sequence of matrices $S(i,j) \in \mathbb{R}^{p\times m}$ such that

$$\mathbf{\Sigma}: \quad u \longmapsto y := \mathcal{S}(u), \;\; y(i) = \sum_{j\in\mathbb{Z}} S(i,j)u(j), \;\; i \in \mathbb{Z} \tag{3.4}$$

This relationship can be written in matrix form as follows

$$\left(\begin{array}{c} \vdots \\ y(-2) \\ y(-1) \\ \hline y(0) \\ y(1) \\ \vdots \end{array}\right) = \left(\begin{array}{ccc|ccc} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & S(-2,-2) & S(-2,-1) & S(-2,0) & S(-2,1) & \cdots \\ \cdots & S(-1,-2) & S(-1,-1) & S(-1,0) & S(-1,1) & \cdots \\ \hline \cdots & S(0,-2) & S(0,-1) & S(0,0) & S(0,1) & \cdots \\ \cdots & S(1,-2) & S(1,-1) & S(1,0) & S(1,1) & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right) \left(\begin{array}{c} \vdots \\ u(-2) \\ u(-1) \\ \hline u(0) \\ u(1) \\ \vdots \end{array}\right)$$

The system described by $\mathcal{S}$ is called *causal* if $S(i,j) = 0,\ i \leq j$, and *time invariant* if $S(i,j) =: S_{i-j} \in \mathbb{R}^{p\times m}$. For a time invariant system $\mathbf{\Sigma}$, we can define the sequence of $p \times m$ constant matrices

$$h_{\mathbf{\Sigma}} = (\ \cdots\ , S_{-2}, S_{-1}, S_0, S_1, S_2,\ \cdots\ )$$

It will be called the *impulse response* of $\mathbf{\Sigma}$ because it is the output obtained in response to a unit pulse $u(t) = \delta(t)$ which is equal to 1, for $t = 0$, and 0 otherwise. Operation (3.4) can now be represented as a *convolution sum*:

$$\mathcal{S}:\ u \mapsto y = \mathcal{S}(u) = h * u \ \text{ where } \ (h*u)(t) = \sum_{k=-\infty}^{\infty} S_{t-k}u(k),\ \ t \in \mathbb{Z} \tag{3.5}$$

Moreover, the matrix representation of $\mathcal{S}$ in this case is a Toeplitz matrix

$$\left(\begin{array}{c} \vdots \\ y(-2) \\ y(-1) \\ \hline y(0) \\ y(1) \\ \vdots \end{array}\right) = \left(\begin{array}{ccc|ccc} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & S_0 & S_{-1} & S_{-2} & S_{-3} & \cdots \\ \cdots & S_1 & S_0 & S_{-1} & S_{-2} & \cdots \\ \hline \cdots & S_2 & S_1 & S_0 & S_{-1} & \cdots \\ \cdots & S_3 & S_2 & S_1 & S_0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right) \left(\begin{array}{c} \vdots \\ u(-2) \\ u(-1) \\ \hline u(0) \\ u(1) \\ \vdots \end{array}\right) \tag{3.6}$$

In the sequel we will restrict our attention to both causal and time-invariant linear systems. The matrix representation of $\mathcal{S}$ in this case is lower triangular and Toeplitz ($S_k = 0,\ k < 0$).

In analogy to the discrete-time case, a *continuous-time linear system* $\mathbf{\Sigma}$, with m input and p output channels can be viewed as a linear operator $\mathcal{S}$ mapping vector-valued functions of dimension m to vector-valued functions of dimension p. In particular we will be concerned with systems which can be expressed by means of an integral

$$\mathcal{S} : u \ \longmapsto\ y,\ y(t) := \int_{-\infty}^{\infty} h(t,\tau)u(\tau)d\tau,\ \ t \in \mathbb{R} \tag{3.7}$$

where $h(t,\tau)$, is a matrix-valued function called the *kernel* or *weighting pattern* of $\mathcal{S}$. The system just defined is *causal* if $h(t,\tau) = 0,\ t \leq \tau$, and *time invariant* if h depends on the difference of the two arguments $h(t,\tau) = h(t-\tau)$. In this case $\mathcal{S}$ is a *convolution operator*

$$\mathcal{S}:\ u \mapsto y = \mathcal{S}(u) = h * u \ \text{ where } \ (h*u)(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau,\ \ t \in \mathbb{R} \tag{3.8}$$

In the sequel we will assume that $\mathcal{S}$ is both causal and time-invariant which means that the upper limit of integration can be replaced by t. In addition, we will assume that h can be expressed as

$$h(t) = S_0\delta(t) + h_a(t),\ S_0 \in \mathbb{R}^{p\times m},\ t \geq 0$$

where δ denotes the δ-distribution and h_a is *analytic*. Hence h_a is uniquely determined by means of the coefficients of its Taylor series expansion at $t = 0^+$:

$$h_a(t) = S_1 + S_2\frac{t}{1!} + S_3\frac{t^2}{2!} + \ \cdots\ + S_k\frac{t^{k+1}}{(k+1)!} + \ \cdots\ ,\ S_k \in \mathbb{R}^{p\times m}$$

It follows that if h can be expressed as above, the output y is at least as smooth as the input u and $\mathbf{\Sigma}$ is consequently called a *smooth* system. Hence just like in the case of discrete-time systems, smooth continuous-time linear system can be described by means of the infinite sequence of $p \times m$ matrices $S_i, i \geq 0$. We formalize this conclusion next.

Definition 3.1 *The* **external representation** *of a time-invariant, causal and smooth continous-time system and that of a time-invariant, causal discrete-time linear system with m inputs and p outputs is given by an infinite sequence of $p \times m$ matrices*

$$h_{\mathbf{\Sigma}} := (S_0,\ S_1,\ S_2,\ \cdots,\ S_k,\ \cdots),\ S_k \in \mathbb{R}^{p\times m} \tag{3.9}$$

The matrices S_k are often referred to as the **Markov parameters** *of the system* $\mathbf{\Sigma}$.

3.1.1 External description in the frequency domain

The (continuous- or discrete-time) Laplace transform of the impulse response (3.20), (3.21) yields the **transfer function** of the system

$$H(\xi) := (\mathcal{L}h)(\xi) \tag{3.10}$$

The Laplace transform is denoted for simplicity by $\mathcal{L}$ for both discrete- and continuous-time, and the Laplace variable ξ is equal to z in the former case and equal to s in the latter. It readily follows that H can be expanded in a formal power series in ξ:

$$H(\xi) = S_0 + S_1\xi^{-1} + S_2\xi^{-2} + \ \cdots\ + S_k\xi^{-k} + \ \cdots$$

This can also be regarded as a Laurent expansion of H around infinity. Consequently (3.5) and (3.8) can be written as follows in the Laplace domain

$$Y(\xi) = H(\xi)U(\xi)$$

An alternative way of describing linear systems *externally* is by specifying a differential or difference equation which relates one of the input and one of the output channels. Given that the input has m and the output p channels. This representation assumes the existence of polynomials $q_{i,j}(\xi), i,j = 1,\cdots,p$ and $p_{i,j}(\xi)$, $i = 1,\cdots,p$, $j = 1,\cdots,m$, such that

$$q_{i,j}\,(\sigma)\,y_j(t) = p_{i,j}\,(\sigma)\,u_i(t)\ \Rightarrow\ Q\,(\sigma)\,y(t) = P\,(\sigma)\,u(t)$$

where P, Q are polynomal matrices $Q \in \mathbb{R}^{p\times p}[\xi]$, $P \in \mathbb{R}^{p\times m}[\xi]$. If we make the assumption that Q is non-singular, that is, its determinant is not identically zero: $\det Q \neq 0$, the transfer function of this system is the rational matrix $H = Q^{-1}P$. If in addition this is *proper rational*, that is, the degree of the numerator of each entry is less than the degree of the corresponding denominator, we can expand this as follows:

$$H(\xi) = Q^{-1}(\xi)P(\xi) = S_0 + S_1\xi^{-1} + \ \cdots\ + \ S_k\xi^{-k}\ + \ \cdots$$

Recall that the variable ξ is used to denote the transform variable s or z, depending on whether we are dealing with continuous- or discrete-time systems. We will not further dwell on this polynomial representation of linear systems since it is the subject of the following contribution in this volume, namely *EOLSS Contribution 6.43.13.5*.

3.1.2 The Bode and Nyquist diagrams

Two important tools in control systems analysis and design are the *Bode diagram* and the *Nyquist diagram*. The former is also known as the *frequency response*. Both are frequency domain tools. Given a continuous-time[2], linear system, with transfer function $L(s)$, the Bode and Nyquist diagrams are depictions of $L(j\omega)$, that is, of the given transfer function evaluated on the imaginary axis.

[2]In this section we will restrict our attention to continous-time systems.

Bode diagram

In the SISO ($m = p = 1$) case, we can express the transfer function in polar form $L(j\omega) = \ell(\omega)e^{j\theta(\omega)}$, where $\ell(\omega)$ is the *amplitude* and $\theta(\omega)$ the *phase* of L at the frequency ω. The *Bode diagram* consists of the *amplitude plot* and of the *phase plot*. The former is a plot of $\ell(j\omega)$ versus ω, and the latter of $\theta(\omega)$ versus ω; more precisely the quantity $20\log_{10}\ell(\omega)$, measured in db (decibels), is plotted vs $\log_{10}\omega$ and $\theta(\omega)$ vs $\log_{10}\omega$. In MATLAB the corresponding command is `bode`.

In the MIMO ($m, p \geq 1$) case, let $L(s)$ be $p \times m$, with $p \geq m$. The *generalized Bode diagram* is composed of the m Bode plots of the singular values $\sigma_i(L(j\omega))$, $i = 1, \cdots, m$, of $L(s)$ evaluated on the imaginary axis. Recall from section 2.2.2 that these are the square roots of the eigenvalues of L times its complex conjugate transpose: $L(j\omega)L^*(-j\omega)$. The resulting m plots depicted on the same diagram are known as the *sigma plot* of the system. In MATLAB the corresponding command is `sigma`. Sometimes instead of the sigma plot, one may depict the SISO Bode plots of each individual entry of the transfer function. This is obtained in MATLAB by using the same command as in the SISO case, namely `bode`.

Nyquist diagram

Again in the SISO ($m = p = 1$) case, the *Nyquist diagram* is a plot in the complex plane of $L(j\omega) \in \mathbb{C}$ as function of the frequency ω which varies from minus infinity to plus infinity. More precisely, the

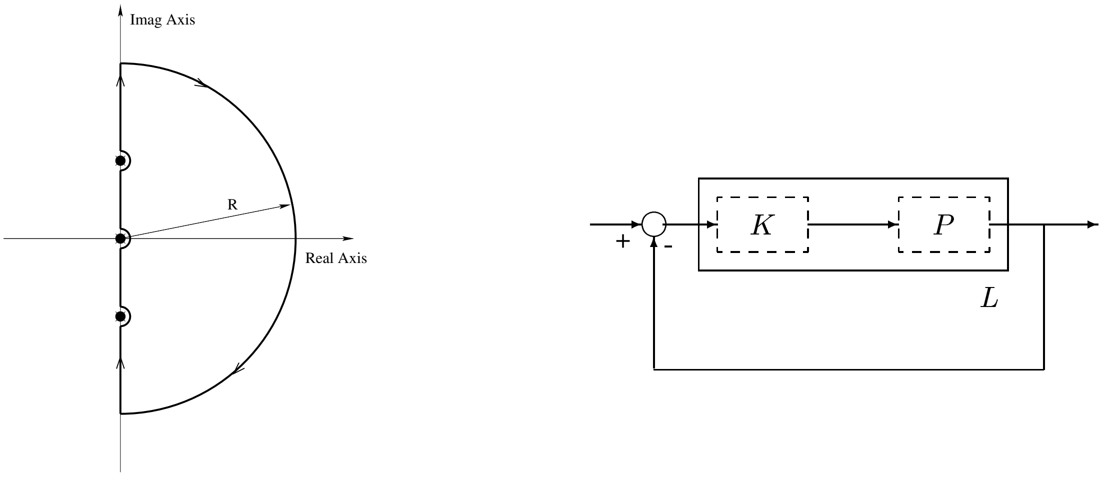


Figure 2: Left: Nyquist contour: $R \longrightarrow \infty$. Right: Unity feedback interconnection

Nyquist diagram is the conformal mapping of the contour shown on the left-hand side of figure 2, by $L(j\omega)$. This contour traverses the imaginary axis from $-\infty$ to $+\infty$ and then goes over to a semicircle in the right-half of the complex plane with radius $R \longrightarrow \infty$. If L has poles on the imaginary axis, the contour has to avoid them, by tracing semicircles in the right-half plane, having radius $\rho \to 0$. Therefore the Nyquist curve is also a closed contour, which however may encircle a particular point of the complex plane several times.

Given the feedback configuration on the right side of figure 2, let $L(s)$ be the open-loop transfer function of the continuous-time feedback system shown (often $L(s) = P(s)K(s)$, where $P(s)$ is the transfer function of the plant and $K(s)$ that of the compensator). The usefulness of the Nyquist diagram lies in the fact that by counting the number of times that this closed curve encircles the point -1 on the real axis of the complex plane, one can determine how many poles the *closed-loop* system possesses in the right-half plane. Equivalently, we may consider the Nyquist diagram of $1 + L$, in which case we will be concerned with encirclements of the origin. Formally, let Γ denote the Nyquist diagram of $1 + L(s)$. We will assume that the feedback loop is well posed, that is $1 + L(\infty) \neq 0$.

The following result is due to Harry Nyquist and dates from the 1930s.

Theorem 3.1 Nyquist criterion: SISO systems.
Consider the system described by the transfer function $L(s)$. Assuming that the Nyquist plot Γ of $1 + L$ does not pass through the origin, the number of unstable closed-loop poles of the unity-feedback system shown on the right-hand side of figure 2, is equal to the sum of
(a) *the number of times Γ encircles the origin in a clockwise direction, plus*
(b) *the number of unstable open-loop poles of L.*

In the MIMO case, assuming that the system is square, i.e. $m = p \geq 1$, there are two ways of defining the *generalized Nyquist* diagram and the above theorem. The most straightforward involves the Nyquist diagram of an associated SISO system, namely

$$\Gamma: \text{ Nyquist plot of } \det\left(I_m + L(s)\right)$$

We will assume that the feedback system shown on the right-hand side of figure 2 is well posed, that is, $\det\left(I + L(\infty)\right) \neq 0$. The following analogous result to theorem 3.1 holds.

Theorem 3.2 Generalized Nyquist criterion: MIMO systems.
Given the square MIMO system $L(s)$, let Γ be the Nyquist plot of $\det\left(I + L(s)\right)$. Assuming that Γ does not pass through the origin, the number of unstable closed-loop poles of the unity-feedback configuration shown on the right-hand side of figure 2, is equal to the sum of
(a) *the number of times Γ encircles the origin in a clockwise direction, plus*
(b) *the number of unstable open-loop poles of L.*

Notice that the number of clockwise encirclements of the origin may be negative. As a matter of fact, the closed loop system (both in the SISO and MIMO cases) is stable if and only if, the number of unstable poles of L is equal to the number of anti-clockwise encirclements of the origin by the Nyquist plot Γ.

A second way of defining the *generalized Nyquist diagram* is as follows. Consider the m eigenvalues $\lambda_i(L(j\omega))$, $i = 1, \cdots, m$, of $L(j\omega)$; clearly, the eigenvalues of $I_m + L(j\omega)$ are $1 + \lambda_i(L(j\omega))$, $i = 1, \cdots, m$. Let Γ_i, $i = 1, \cdots, m$, be the associated Nyquist plots. In this case, the Γ_i need not be continuous functions of ω, and may have bifurcation or branch points. However, it can be shown that after all eigenvalues have been plotted, m *continuous* and *closed* curves can be identified; we will call these $\hat{\Gamma}_i$, $i = 1, \cdots, m$. These curves can be considered as a generalization of the Nyquist diagram in the MIMO case.

An appropriately modified version of theorem 3.2 can be stated as follows. The determinant of $I_m + L(j\omega)$ is equal to the product of these eigenvalues. Thus the number of encirclements of the origin by Γ is equal to the sum of the number of encirclements of $\hat{\Gamma}_i$, $i = 1, \cdots, m$. It follows that under the assumptions stated, the number of unstable closed-loop poles is equal to the number of unstable open-loop poles plus the total number of clockwise encirclements of the origin by the $\hat{\Gamma}_i$, $i = 1, \cdots, m$.

Example 3.1 Consider the 2-input, 2-output system with transfer function:

$$L_k(s) = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s+2} \\ 1 & \frac{k}{s+3} \end{bmatrix}, \ k \in \mathbb{R},$$

which depends on the real parameter k. A state space realization of $L_k(s)$ is computed in example 3.6. We will first draw the generalized Bode or sigma plot. The quantities of interest are the square roots of the eigenvalues of the following matrix

$$L_k(j\omega)L_k^*(-j\omega) = \begin{bmatrix} \frac{2\,\omega^2+5}{\omega^4+5\,\omega^2+4} & \frac{(\omega^2-k+6)+j(1+k)\omega}{-(2\omega^2+6)+j\omega(\omega^2+5)} \\ \frac{-(\omega^2-k+6)+j(1+k)\omega}{(2\omega^2+6)+j\omega(\omega^2+5)} & \frac{\omega^2+9+k^2}{\omega^2+9} \end{bmatrix} \geq 0$$

$\sigma_1(L_k(j\omega)) = \sqrt{\lambda_1(L_k(j\omega)L_k^*(-j\omega))}$, and $\sigma_2(L_k(j\omega)) = \sqrt{\lambda_2(L_k(j\omega)L_k^*(-j\omega))}$. These quantities are shown in figure 3. Next we turn our attention to the generalized Nyquist plots. The eigenvalues and the

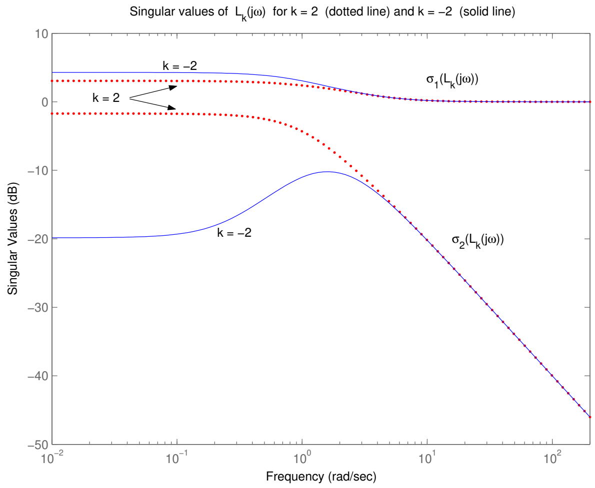


Figure 3: Generalized Bode diagrams of $L_k(s)$. Solid curves: $k = -2$; dotted curves: $k = 2$. Top curves: $\sigma_1(L_k(j\omega))$; bottom curves: $\sigma_2(L_k(j\omega))$.

determinant of $I_2 + L_k(s)$ are:

$$\lambda_{1,2} = \frac{2s^3 + 11s^2 + 9s - 10 \pm \sqrt{(s+2)(4s^4 + 17s^3 - 16s^2 - 43s + 86)}}{2(s-1)(s+3)(s+2)}, \; k = 2$$

$$\lambda_{1,2} = \frac{2s^3 + 7s^2 + 5s - 2 \pm \sqrt{(s+2)(4s^4 + 25s^3 + 16s^2 - 35s + 38)}}{2(s-1)(s+3)(s+2)}, \; k = -2$$

$$\det\,(I_2 + L_k(s)) = \frac{s^3 + (k+4)s^2 + (2k+4)s + 3}{(s-1)(s+3)(s+2)}.$$

The closed-loop transfer function of the configuration shown in figure 2 is

$$L_k(I_2 + L_k)^{-1} = \begin{bmatrix} \frac{(3+k)s+9+2\,k}{s^3+(k+4)s^2+(2\,k+4)s+3} & \frac{(s-1)(s+3)}{s^3+(k+4)s^2+(2\,k+4)s+3} \\ \frac{(s-1)(s+3)(s+2)}{s^3+(k+4)s^2+(2\,k+4)s+3} & \frac{(k-1)s^2+(2\,k-2)s+3}{s^3+(k+4)s^2+(2\,k+4)s+3} \end{bmatrix}.$$

The characteristic polynomial of the closed loop is equal to $\chi_{\rm CL}(s) = s^3+(k+4)s^2+(2k+4)s+3$. A simple root-locus argument shows that the closed-loop system is stable for all values of the parameter $k \leq k_0 = -3 + \frac{\sqrt{10}}{2}$; for $k = k_0$ the characteristic polynomial has a pair of pure imaginary roots $\pm j(\sqrt{10} - 2)$; and for $k > k_0$ the

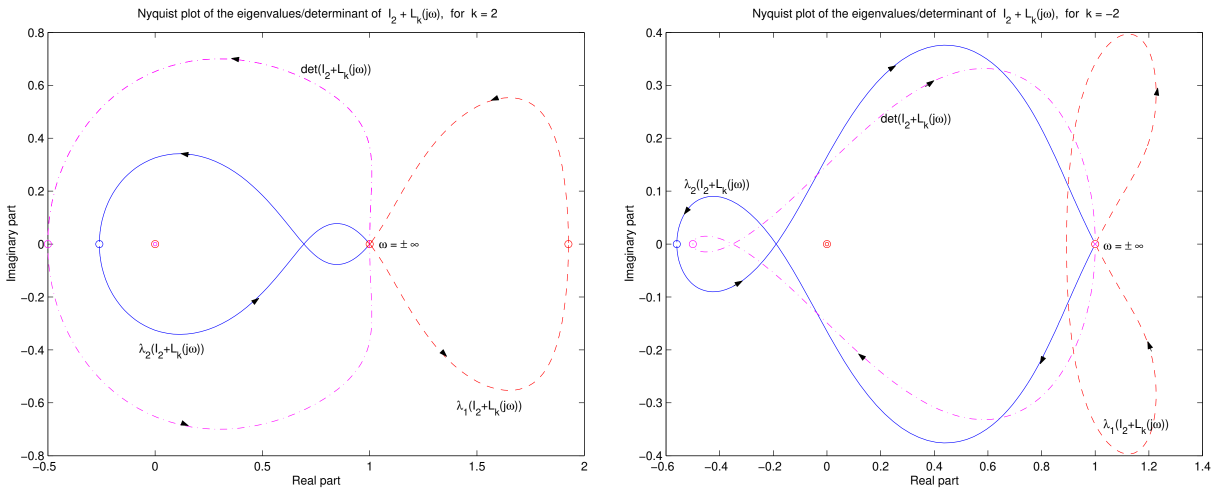


Figure 4: Generalized Nyquist diagrams of $I_2 + L_k(s)$. Left-hand plots: $k = 2$. Right-hand plots: $k = -2$. Dashed curves: $\lambda_1(I_2 + L_k(j\omega))$. Solid curves: $\lambda_2(I_2 + L_k(j\omega))$. Dash-dot curves: $\det(I_2 + L_k(j\omega))$. Arrows show direction of increasing frequency.

closed loop system becomes unstable. In particular for $k = k_1 = 2 > k_0$, the closed-loop system is stable, while for $k = k_2 = -2 < k_0$, it is unstable.

Notice that the open-loop system has an unstable pole ($s = 1$). Thus for the closed-loop system to be stable the Nyquist plot *must* encircle the origin *once* in the anti-clockwise direction. As shown in the left-hand plot of figure 4, for $k = 2$, this is indeed the case. The right-hand plot of the same figure, however, shows that for $k = -2$, there is a single clockwise encirclement of the origin and thus the closed loop system, according to theorem 3.2, has 2 unstable poles. Indeed, the roots of χ_{CL} are -2.4856 and $0.2428 \pm 1.0715j$. ■

3.2 Internal representation

An alternative description for linear systems is the *internal representation* which uses in addition to the input u and the output y, the state x. For our purposes, given are three linear finite-dimensional spaces. The first one is the *state space* $X \cong \mathbb{K}^n$ (the symbol $\cong$ denotes *isomorphism*, that is, X is a linear space which is isomorphic to the n-dimensional space $\mathbb{K}^n$; as an example the space X of all polynomials of degree less than n is isomorphic to $\mathbb{R}^n$, since there is a one-to-one correspondence between each polynomial and an n-vector consisting of its coefficients). The second space is the *input space* $U \cong \mathbb{K}^m$, and the *output space* $Y \cong \mathbb{K}^p$ (recall that $\mathbb{K}$ denotes the field of real numbers $\mathbb{R}$ or that of complex numbers $\mathbb{C}$). The *state equations* describing a linear system are a set of first order linear *differential* or *difference* equations, according to whether we are dealing with a continuous- or a discrete-time system:

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t),\ t \in \mathbb{R} \text{ or} \tag{3.11}$$

$$x(t+1) = Ax(t) + Bu(t),\ t \in \mathbb{Z} \tag{3.12}$$

In both cases $x(t) \in X$ is the *state* of the system at time t, while $u(t) \in U$, is the value of the input function at time t. Moreover, $B : U \longrightarrow X$, $A : X \longrightarrow X$, are linear maps; the first one is called the *input* map, while the second one describes the *dynamics* or *internal evolution* of the system. Equations (3.11) and (3.12) can be written in a unified way as follows:

$$\sigma x = Ax + Bu \tag{3.13}$$

where σ denotes the derivative operator for continuous-time systems, and the (backwards) shift operator for discrete-time systems.

The *output equations*, for both discrete- and continuous-time linear systems, are composed of a set of linear algebraic equations

$$y = Cx + Du \tag{3.14}$$

where y is the output function (response), and $C : X \longrightarrow Y$, $D : U \longrightarrow Y$, are linear maps; C is called the *output* map. It describes how the system interacts with the outside world.

In the sequel the term *linear system* $\mathbf{\Sigma}$ in internal representation will be used to denote a linear, time-invariant, continuous- or discrete-time system which is finite-dimensional. Linear means: U, X, Y are linear spaces, and A, B, C, D are linear maps; finite-dimensional means: U, X, Y are all finite dimensional; time-invariant means: A, B, C, D do not depend on time; their matrix representations are constant $n \times n$, $n \times m$, $p \times n$, $p \times m$ matrices. In the sequel (by slight abuse of notation) we will denote the linear maps A, B, C, D as well as their matrix representations (in some appropriate basis) with the same symbols. We are now ready to give the

Definition 3.2 **(a)** *A* **linear system** *in* **internal** *or* **state space** *representation is a quadruple of linear maps (matrices)*

$$\mathbf{\Sigma} := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right),\ A \in \mathbb{K}^{n\times n},\ B \in \mathbb{K}^{n\times m},\ C \in \mathbb{K}^{p\times n},\ D \in \mathbb{K}^{p\times m} \tag{3.15}$$

The **dimension** *of the system is defined as the dimension of the associated state space:*

$$\dim \mathbf{\Sigma} = n \tag{3.16}$$

(b) $\mathbf{\Sigma}$ *is called* **stable** *if the eigenvalues of A have negative real parts or lie inside the unit disc, depending on whether $\mathbf{\Sigma}$ is a continuous-time or a discrete-time system.*

3.2.1 Solution in the time domain

Let $\phi(u; x_0; t)$ denote the solution of the state equations (3.13), i.e., the state of the system at time t attained from the initial state x_0 at time t_0, under the influence of the input u. In particular, for the continuous-time state equations (3.11)

$$\phi(u; x_0; t) \ = \ e^{A(t-t_0)}x_0 \ + \ \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau,\ \ t \ \geq \ t_0, \tag{3.17}$$

while for the discrete-time state equations (3.12)

$$\phi(u; x_0; t) \ = \ A^{t-t_0}x_0 \ + \ \sum_{j=t_0}^{t-1} A^{t-1-j}Bu(j),\ \ t \ \geq \ t_0. \tag{3.18}$$

In the above formulae we may assume without loss of generality, that $t_0 = 0$, since the systems we are dealing with are time-invariant. The first summand in the above expressions is called *zero input* and the second *zero state* part of the solution. The nomenclature comes from the fact that the zero input part is obtained when the system is excited exclusively by means of initial conditions and the zero state part is the result of excitation by some input u and zero initial conditions. In the tables that follow these parts are denoted with the subscripts ”zi” and ”zs”.

For both discrete- and continuous-time systems it follows that the output is given by:

$$y(t) = C\phi(u; x(0); t) + Du(t) = C\phi(0; x(0); t) + C\phi(u; 0; t) + Du(t) \tag{3.19}$$

Again the same remark concerning the *zero-input* and the *zero state* parts of the output holds.

If we compare the above expressions for $t_0 = -\infty$ and $x_0 = 0$, with (3.5) and (3.8) it follows that the *impulse response* h has the form below. For continuous-time systems:

$$h(t) := \begin{cases} Ce^{At}B + \delta(t)D, \ t \geq 0 \\ 0, \ t < 0 \end{cases} \tag{3.20}$$

where δ denotes the δ-distribution. For discrete-time systems

$$h(t) := \begin{cases} CA^{t-1}B, \ t > 0 \\ D, \ t = 0 \\ 0, \ t < 0 \end{cases} \tag{3.21}$$

The corresponding external representation given by means of the Markov parameters (3.9), is:

$$h_{\mathbf{\Sigma}} = (D,\ CB,\ CAB,\ CA^2B,\ \cdots,\ CA^{k-1}B,\ \cdots) \tag{3.22}$$

By transforming the state the matrices which describe the system will change. Thus, if the new state is $\tilde{x} := Tx$, $\det T \neq 0$, (3.13) and (3.14) in the new state $\tilde{x}$, will be become

$$\sigma\tilde{x} = \underbrace{TAT^{-1}}_{\tilde{A}}\tilde{x} + \underbrace{TB}_{\tilde{B}}u, \ \ y = \underbrace{CT^{-1}}_{\tilde{C}}\tilde{x} + Du$$

where D remains unchanged. The corresponding triples are called *equivalent*. Put differently, $\mathbf{\Sigma}$ and $\tilde{\mathbf{\Sigma}}$ are equivalent if there exists T such that:

$$\left(\begin{array}{c|c} T & \\ \hline & I_p \end{array}\right)\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right)\left(\begin{array}{c|c} T & \\ \hline & I_m \end{array}\right), \ \det T \neq 0 \tag{3.23}$$

Let $\mathbf{\Sigma}$ and $\tilde{\mathbf{\Sigma}}$ be equivalent with equivalence transformation T. It readily follows that

$$\begin{aligned} H_{\mathbf{\Sigma}}(\xi) &= D + C(\xi I - A)^{-1}B = D + CT^{-1}T(\xi I - A)^{-1}T^{-1}TB \\ &= D + CT^{-1}(\xi I - TAT^{-1})^{-1}TB = \tilde{D} + \tilde{C}(\xi I - \tilde{A})^{-1}\tilde{B} = H_{\tilde{\mathbf{\Sigma}}}(\xi) \end{aligned}$$

This immediately implies that $S_k = \tilde{S}_k$, $k \in \mathbb{N}$. We have thus proved

Proposition 3.1 *Equivalent triples have the same transfer function and therefore the same Markov parameters.*

3.2.2 Solution in the frequency domain

In this section we will assume that the initial time is $t_0 = 0$. Let $\Phi(\xi) = \mathcal{L}(\phi)(\xi)$, where ϕ is defined by (3.17), (3.18); there holds

$$\Phi(\xi) = (\xi I - A)^{-1}x_0 + (\xi I - A)^{-1}BU(\xi) \ \Rightarrow \ Y(\xi) = C\Phi(\xi) + DU(\xi)$$

Thus, by (3.10), (3.20), (3.21), the *transfer function* of $\mathbf{\Sigma}$ is:

$$H_{\mathbf{\Sigma}}(\xi) = D + C(\xi I - A)^{-1}B \tag{3.24}$$

A summary of these relationships are provided in the table that follows.

3.2.3 The concepts of reachability and observability

The concept of *reachability* provides the tool for answering questions related to the extend to which the state of the system x can be manipulated through the input u. The related concept of *controllability* will be discussed subsequently. Both concepts involve only the state equations.

Definition 3.3 *Given is* $\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline & \end{array}\right)$, $A \in \mathbb{K}^{nxn}$, $B \in \mathbb{K}^{nxm}$. *A state* $\bar{x} \in X$ *is* **reachable** *from the zero state if there exist an input function* $\bar{u}(t)$ *and a time* $\bar{T} < \infty$, *such that* $\bar{x} = \phi(\bar{u}; 0; \bar{T})$. *The* **reachable subspace** $X^{reach} \subset X$ *of* $\mathbf{\Sigma}$, *is the set which contains all reachable states of* $\mathbf{\Sigma}$. *We will call the system* $\mathbf{\Sigma}$ **(completely) reachable** *if* $X^{reach} = X$. *Furthermore*

$$\mathcal{R}_n(A, B) \;:=\; [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B] \tag{3.25}$$

will be called the **reachability matrix** *of* $\mathbf{\Sigma}$. *The finite* **reachability gramians** *at time* $t < \infty$ *are defined as follows. For continuous-time systems:*

$$\mathcal{P}(t) := \int_0^t e^{A\tau} BB^* e^{A^*\tau} d\tau, \; t \in \mathbb{R}_+ \tag{3.26}$$

while for discrete-time systems

$$\mathcal{P}(t) := \mathcal{R}_t(A, B)\mathcal{R}_t^*(A, B) = \sum_{k=0}^{t-1} A^k BB^*(A^*)^k, \; t \in \mathbb{Z}_+ \tag{3.27}$$

Theorem 3.3 *Consider the pair* (A, B) *as defined above.* **(a)** $X^{reach} = \operatorname{span}\operatorname{col}\mathcal{R}_n = \operatorname{span}\operatorname{col}\mathcal{P}(t)$, *where* $t > 0$, $t \geq n-1$, *for continuous-, discrete-time systems, respectively.* **(b) Reachability conditions**. *The following are equivalent:*

1. The pair (A, B), $A \in \mathbb{K}^{nxn}$, $B \in \mathbb{K}^{nxm}$, is completely reachable.
2. The rank of the reachability matrix is full: $\operatorname{rank}\mathcal{R}(A, B) = n$.
3. The reachability gramian is positive definite $\mathcal{P}(t) > 0$, for some $t > 0$.
4. No left eigenvector v^* of A is in the left kernel of B: $v^*A = \lambda v^* \;\;\Rightarrow\;\; v^*B \neq 0$.
5. $\operatorname{rank}(\mu I_n - A \quad -B) = n$, for all $\mu \in \mathbb{C}$.
6. The polynomial matrices $\xi I - A$ and B are left coprime.

The fourth, fifth, and sixth conditions in the theorem above are known as the **PHB** or **Popov-Hautus-Belevich** tests for reachability.

We now turn our attention to the concept of *observability*. In order to be able to modify the dynamical behavior of a system, very often the state x needs to be available. Typically however the state variables are inaccessible and only certain linear combinations y thereof, given by the output equations (3.14) are known. Thus we need to discuss the problem of reconstructing the state $x(T)$ from observations $y(\tau)$ where τ is in some appropriate interval. If $\tau \in [T, T+t]$ we have the *state observation problem*, while if $\tau \in [T-t, T]$ we have the *state reconstruction problem*.

We will first discuss the observation problem. Without loss of generality we will assume that $T = 0$. Recall (3.17), (3.18) and (3.19). Since the input u is known, the latter two terms in (3.19) are also known for $t \geq 0$. Therefore, in determining $x(0)$ we may assume without loss of generality that $u(\cdot) = 0$. Thus, the observation problem reduces to the following: given $C\phi(0;x(0);t)$ for $t \geq 0$, find $x(0)$. Since B and D are irrelevant, for this subsection

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & \\ \hline C & \end{array}\right), \; A \in \mathbb{K}^{nxn}, \; C \in \mathbb{K}^{pxn}$$

Definition 3.4 *A state $\bar{x} \in X$ is* **unobservable** *if $y(t) = C\phi(0;\bar{x};t) = 0$, for all $t \geq 0$, i.e. if $\bar{x}$ is indistinguishable from the zero state for all $t \geq 0$. The* **unobservable subspace** *X^{unobs} of X is the set of all unobservable states of $\mathbf{\Sigma}$. $\mathbf{\Sigma}$ is (completely)* **observable** *if $X^{unobs} = 0$. The* **observability matrix** *of $\mathbf{\Sigma}$ is*

$$\mathcal{O}_n(C,A) \;=\; (C^* \;\; A^*C^* \;\; \cdots \;\; (A^*)^{n-1}C^*)^* \tag{3.28}$$

The finite **observability gramians** *at time $t < \infty$ are:*

$$\mathcal{Q}(t) \;:=\; \int_0^t e^{A^*(t-\tau)}C^*Ce^{A(t-\tau)}d\tau, \; t \in \mathbb{R}_+ \tag{3.29}$$

$$\mathcal{Q}(t) := \mathcal{O}_t^*(C,A)\mathcal{O}_t(C,A), \; t \in \mathbb{Z}_+ \tag{3.30}$$

Theorem 3.4 *Given $\mathbf{\Sigma} = \left(\begin{array}{c|c} A & \\ \hline C & \end{array}\right)$, for both $t \in \mathbb{Z}$ and $t \in \mathbb{R}$, X^{unobs} is a linear subspace of X given by*

$$X^{unobs} = \ker \mathcal{O}_n(C,A) = \ker \mathcal{Q}(t) = \{x \in X : \; CA^{i-1}x = 0, \; i > 0\}$$

where $t > 0$, $t \geq n-1$, depending on whether the system is continuous-, or discrete-time. Thus, $\mathbf{\Sigma}$ is completely observable if, and only if, $\operatorname{rank} \mathcal{O}(C,A) = n$.

Remark 3.1 **(a)** Given $y(t), \; t \geq 0$, let Y_0 denote the following $np \times 1$ vector:

$$Y_0 := (y^*(0) \;\; y^*(1) \;\; \cdots \;\; y^*(n-1))^*, \; t \in \mathbb{Z}, \;\; Y_0 := (y^*(0) \;\; Dy^*(0) \;\; \cdots \;\; D^{n-1}y^*(0))^*, \; t \in \mathbb{R},$$

where $D := \frac{d}{dt}$. The observation problem reduces to the solution of the linear set of equations $\mathcal{O}_n(C,A)x(0) = Y_0$. This set of equations is solvable for all initial conditions $x(0)$, i.e. it has a *unique* solution if and only if $\mathbf{\Sigma}$ is observable. Otherwise $x(0)$ can only be determined modulo X^{unobs}, i.e. up to an arbitrary linear combination of unobservable states.

(b) If $x_1, \; x_2$, are not reachable, there is a trajectory passing through the two points if, and only if, $x_2 - f(A,T)x_1 \in X^{reach}$, for some T, where $f(A,T) = e^{AT}$ for continuous-time systems and $f(A,T) = A^T$ for discrete-time systems. This shows that if we start from a reachable state $x_1 \neq 0$ the states that can be attained are also within the reachable subspace. ■

A concept which is closely related to reachability is that of *controllability*. Here, instead of driving the zero state to a desired state, a given non-zero state is steered to the zero state. Furthermore, a state $\bar{x} \in X$ is *unreconstructible* if $y(t) = C\phi(0;\bar{x};t) = 0$, for all $t \leq 0$, i.e. if $\bar{x}$ is indistinguishable from the zero state for all $t \leq 0$.

The next result shows that for continuous-time systems the concepts of reachability and controllability are equivalent while for discrete-time systems the latter is weaker. Similarly, while for continuous-time systems the concepts of observability and reconstructibility are equivalent, for discrete-time systems the latter is weaker. For this reason, only the concepts of reachability and observability are used in the sequel.

Proposition 3.2 *Given is the triple (C,A,B).* (a) *For continuous-time systems $X^{contr} = X^{reach}$ and $X^{unrec} = X^{unobs}$.* (b) *For discrete-time systems $X^{reach} \subset X^{contr}$ and $X^{unrec} \supset X^{unobs}$; in particular $X^{contr} = X^{reach} + \ker A^n$ and $X^{unrec} = X^{unobs} \cap \operatorname{im} A^n$.*

3.2.4 The infinite gramians

Consider a continuous-time linear system $\mathbf{\Sigma}_c = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ which is *stable*, i.e. all eigenvalues of A have *negative real parts*. In this case both (3.26) as well as (3.29) are defined for $t = \infty$. In addition because of Plancherel's formula, the gramians can be expressed also in the frequency domain (expressions on the right-hand side):

$$\mathcal{P} := \int_0^\infty e^{A\tau} BB^* e^{A^*\tau} d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega - A)^{-1} BB^* (-j\omega - A^*)^{-1} d\omega \tag{3.31}$$

$$\mathcal{Q} := \int_0^\infty e^{A^*\tau} C^* C e^{A\tau} d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-j\omega - A^*)^{-1} C^* C (j\omega - A)^{-1} d\omega \tag{3.32}$$

$\mathcal{P}$, $\mathcal{Q}$ are the *infinite reachability* and *infinite observability gramians* associated with $\mathbf{\Sigma}_c$. These gramians satisfy the following linear matrix equations, called *Lyapunov equations*.

Proposition 3.3 *Given the stable, continuous-time system $\mathbf{\Sigma}_c$ as above, the associated infinite reachability gramian $\mathcal{P}$ satisfies the continuous-time Lyapunov equation*

$$A\mathcal{P} + \mathcal{P}A^* + BB^* = 0 \tag{3.33}$$

while the associated infinite observability gramian satisfies

$$A^*\mathcal{Q} + \mathcal{Q}A + C^*C = 0 \tag{3.34}$$

Proof. Due to stability

$$A\mathcal{P} + \mathcal{P}A^* = \int_0^\infty Ae^{A\tau} BB^* e^{A^*\tau} A^* d\tau = \int_0^\infty d(e^{A\tau} BB^* e^{A^*\tau}) = -BB^*$$

This proves (3.33); (3.34) is proved similarly. ■

If the discrete-time system $\mathbf{\Sigma}_d = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is *stable*, i.e. all eigenvalues of A are inside the unit disc, the gramians (3.27) as well as (3.30) are defined for $t = \infty$

$$\mathcal{P} := \mathcal{R}(A,B)\mathcal{R}(A,B)^* = \sum_{t>0} A^{t-1} BB^* (A^*)^{t-1} = \frac{1}{2\pi} \int_0^{2\pi} (e^{j\theta} I - A)^{-1} BB^* (e^{-j\theta} I - A^*)^{-1} d\theta \tag{3.35}$$

$$\mathcal{Q} := \mathcal{O}(C,A)^*\mathcal{O}(C,A) = \sum_{t>0} (A^*)^{t-1} C^* C A^{t-1} = \frac{1}{2\pi} \int_0^{2\pi} (e^{-j\theta} I - A^*)^{-1} C^* C (e^{j\theta} I - A)^{-1} d\theta \tag{3.36}$$

Notice that $\mathcal{P}$ can be written as $\mathcal{P} = BB^* + A\mathcal{P}A^*$; moreover $\mathcal{Q} = C^*C + A^*\mathcal{Q}A$. These are the so-called **discrete Lyapunov** or **Stein** equations:

Proposition 3.4 *Given the stable, discrete-time system $\mathbf{\Sigma}_d$ as above, the associated infinite reachability gramian $\mathcal{P}$ satisfies the while the associated infinite observability gramian $\mathcal{Q}$ satisfies discrete-time Lyapunov equation*

$$A\mathcal{P}A^* + BB^* = \mathcal{P}, \; A^*\mathcal{Q}A + B^*B = \mathcal{Q} \tag{3.37}$$

We conclude this section by summarizing some properties of the system gramians.

Lemma 3.1 *Let $\mathcal{P}$ and $\mathcal{Q}$ denote the infinite gramians of a linear stable system.*

(a) *The minimal energy required to steer the state of the system from* 0 *to* x_r *is* $x_r^* \mathcal{P}^{-1} x_r$.

(b) *The maximal energy produced by observing the output of the system whose initial state is* x_o *is* $x_o^* \mathcal{Q} x_o$.

(c) *The states which are difficult, i.e. require large amounts of energy, to reach are in the span of those eigenvectors of $\mathcal{P}$ which correspond to small eigenvalues. Furthermore, the states which are difficult to observe, i.e. produce small observation energy, are in the span of those eigenvectors of $\mathcal{Q}$ which correspond to small eigenvalues.*

Remark 3.2 *Computation of the reachability gramian.* Given the pair $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, the reachability gramian is defined by (3.26). We will assume that the eigenvalues of A are distinct. Then A is diagonalizable; let the EVD (Eigenvalue Decomposition) be

$$A = V\Lambda V^{-1} \text{ where } V = [v_1 \;\; v_2 \;\; \cdots \;\; v_n],\; \Lambda = \operatorname{diag}(\lambda_1, \; \cdots \; , \lambda_n)$$

v_i denotes the eigenvector corresponding to the eigenvalue λ_i. Notice that if the i$^{\text{th}}$ eigenvalue is complex, the corresponding eigenvector will also be complex. Let

$$W = V^{-1} B \in \mathbb{C}^{n\times m}$$

and denote by $W_i \in \mathbb{C}^{1\times m}$ the i$^{\text{th}}$ row of W. With the notation introduced above the following formula holds:

$$\mathcal{P}(T) = V\mathcal{R}(T)V^* \quad \text{where} \quad [\mathcal{R}(T)]_{ij} = \frac{-W_i W_j^*}{\lambda_i + \lambda_j^*}\Big(1 - \exp\left[(\lambda_i + \lambda_j^*)T\right]\Big) \in \mathbb{C}$$

Furthermore, if $\lambda_i + \lambda_j^* = 0$, $[\mathcal{R}(T)]_{ij} = (W_i W_j^*)\, T$. If in addition A is stable, the infinite gramian (3.31) is given by $\mathcal{P} = V\mathcal{R}V^*$, where $\mathcal{R}_{ij} = \frac{-W_i W_j^*}{\lambda_i + \lambda_j^*}$. This formula accomplishes both the computation of the exponential and the integration explicitly, in terms of the EVD of A. ■

Example 3.2 Consider the example of the parallel connection of two branches, the first consisting of the series connection of an inductor L with a resistor R_L, and the other consisting of the series connection of a capacitor C with a resistor R_C. Assume that the values of these elements are $L = 1$, $R_L = 1$, $C = 1$, $R_C = \frac{1}{2}$; then

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},\; B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \;\Rightarrow\; e^{At}B = \begin{bmatrix} e^{-t} \\ 2e^{-2t} \end{bmatrix}$$

The gramian $\mathcal{P}(T)$ and the infinite gramian $\mathcal{P}$ are:

$$\mathcal{P}(T) = \begin{bmatrix} -1/2\, e^{-2T} + 1/2 & -2/3\, e^{-3T} + 2/3 \\ -2/3\, e^{-3T} + 2/3 & -e^{-4T} + 1 \end{bmatrix},\; \mathcal{P} = \lim_{T\to\infty} \mathcal{P}(T) = \begin{bmatrix} 1/2 & 2/3 \\ 2/3 & 1 \end{bmatrix}$$

If the system is asymptotically stable, i.e. $\mathcal{R}e(\lambda_i(A)) < 0$, the reachability gramian is defined for $T = \infty$, and it satisfies (3.33). Hence, the infinite gramian can be computed as the solution to the above linear matrix equation; no explicit calculation of the matrix exponentials, mutliplication and subsequent intergration is required. In matlab, if in addition the pair (A, B) is controllable, we have:

$$\mathcal{P} = \text{lyap}\,(A, B * B')$$

For the matrices defined earlier, using the `'lyap'` command in the format `'long e'`, we get:

$$\mathcal{P} = \begin{bmatrix} 5.000000000000000e-01 & 6.666666666666666e-01 \\ 6.666666666666666e-01 & 1.000000000000000e+00 \end{bmatrix}$$

■

Example 3.3 A second simple example is the following:

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad e^{At} = \begin{pmatrix} -e^{-2t} + 2e^{-t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & 2e^{-2t} - e^{-t} \end{pmatrix}$$

This implies

$$\mathcal{P}(T) = \frac{1}{12} \begin{pmatrix} -6e^{-2T} + 8e^{-3T} - 3e^{-4T} + 1 & -12e^{-3T} + 6e^{-2T} + 6e^{-4T} \\ -12e^{-3T} + 6e^{-2T} + 6e^{-4T} & -12e^{-4T} + 16e^{-3T} - 6e^{-2T} + 2 \end{pmatrix}$$

And finally

$$\mathcal{P} = \texttt{lyap}\,(A, B * B') = \begin{pmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}$$

■

A transformation between continuous and discrete time systems

One transformation between continuous- and discrete-time systems is given by the bilinear transformation of the complex plane onto itself given by $z = \frac{1+s}{1-s}$. In particular, the transfer function $H_c(s)$ of a continuous-time system is obtained from that of a discrete-time one $H_d(z)$ as follows:

$$H_c(s) = H_d\left(\frac{1+s}{1-s}\right)$$

This transformation maps the left-half of the complex plane onto the unit disc and vice-versa. The matrices

$$\mathbf{\Sigma}_c := \left(\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right),\quad \mathbf{\Sigma}_d := \left(\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right)$$

of these two systems are related as given in the following table.

Continuous-time		Discrete-time
$A_c,\ B_c,\ C_c,\ D_c$	$z = \frac{1+s}{1-s}$	$\left\{\begin{array}{l} A_d = (I + A_c)(I - A_c)^{-1} \\ B_d = \sqrt{2}(I - A_c)^{-1}B_c \\ C_d = \sqrt{2}C_c(I - A_c)^{-1} \\ D_d = D_c + C_c(I - A_c)^{-1}B_c \end{array}\right.$
$\left.\begin{array}{l} A_c = (A_d + I)^{-1}(A_d - I) \\ B_c = \sqrt{2}(A_d + I)^{-1}B_d \\ C_c = \sqrt{2}C_d(A_d + I)^{-1} \\ D_c = D_d - C_d(A_d + I)^{-1}B_d \end{array}\right\}$	$s = \frac{z-1}{z+1}$	$A_d,\ B_d,\ C_d,\ D_d$

Proposition 3.5 *Given the stable continuous-time system $\mathbf{\Sigma}_c$ with infinite gramians $\mathcal{P}_c$, $\mathcal{Q}_c$, let $\mathbf{\Sigma}_d$ with infinite gramians $\mathcal{P}_d$, $\mathcal{Q}_d$, be the discrete-time system obtained by means of the transformation given above. It follows that the bilinear transformation introduced above preserves the gramians: $\mathcal{P}_c = \mathcal{P}_d$ and $\mathcal{Q}_c = \mathcal{Q}_d$. Furthermore, this transformation preserves the infinity norms (see section 4.3).*

3.3 The realization problem

In the preceding sections we have presented two ways of representing linear systems: the **internal** and the **external**. The former makes use of the inputs u, states x, and outputs y. The latter makes use *only* of the inputs u and the outputs y. The question thus arises as to the relationship between these two representations.

In one direction this problem is trivial. Given the internal representation $\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ of a system, the external representation is readily derived. As shown earlier, the transfer function of the system is given by (3.24) $H_{\mathbf{\Sigma}}(\xi) = D + C(\xi I - A)^{-1}B$, while from (3.22), the Markov parameters are given by

$$S_0 = D,\ S_k := CA^{k-1}B \in \mathbb{R}^{p\times m},\ k = 1,\ 2,\ \cdots . \tag{3.38}$$

The converse problem, i.e. given the external representation, derive the internal one, is far from trivial. This is the **realization problem**: given the external representation of a linear system *construct* an internal or state variable representation. In other words, given the impulse response h, or equivalently the transfer function H, or the Markov parameters S_k of a system, construct $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$, such that (3.38) hold. It readily follows without computation that $D = S_0$. Hence the following problem results:

Definition 3.5 *Given the sequence of $p \times m$ matrices S_k, $k = 1, 2, \cdots$, the* **realization problem** *consists in finding a positive integer n and constant matrices (C, A, B) such that*

$$S_k = CA^{k-1}B,\ C, A, B \in \mathbb{R}^{p\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m},\ k = 1,\ 2,\ \cdots . \tag{3.39}$$

The triple $\left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$ *is then called a* **realization** *of the sequence S_k, and the latter is called a* **realizable** *sequence.*

The realization problem is sometimes referred to as the **problem of construction of state** for linear systems described by convolution relationships.

Remark 3.3 Realization can also be considered as the problem of converting frequency domain data into time domain data. The reason is that measurement of the Markov parameters is closely related to measurement of the frequency response. ∎

Example 3.4 Consider the following (scalar) sequences:

$$\begin{aligned}
(S_k)_{k=0}^{\infty} &= \{1, 1, 1, 1, 1, 1, 1, 1, 1,\ \cdots\} \\
(S_k)_{k=0}^{\infty} &= \{1, 2, 3, 4, 5, 6, 7, 8, 9,\ \cdots\}\ \text{natural numbers} \\
(S_k)_{k=0}^{\infty} &= \{1, 2, 3, 5, 8, 13, 21, 34, 55,\ \cdots\}\ \text{Fibonacci numbers} \\
(S_k)_{k=0}^{\infty} &= \{1, 2, 3, 5, 7, 11, 13, 17, 19,\ \cdots\}\ \text{primes} \\
(S_k)_{k=0}^{\infty} &= \{1, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \frac{1}{5!}, \frac{1}{6!},\ \cdots\}\ \text{inverse factorials}
\end{aligned}$$

Which sequences are realizable? ∎

Problem 3.1 The following problems arise:

(a) *Existence*: given a sequence S_k, $k > 0$, determine whether there exist a positive integer n and a triple of matrices A, B, C such that (3.39) holds.
(b) *Uniqueness*: in case such an integer and triple exist, are they unique in some sense?
(c) *Construction*: in case of existence, find n and give an algorithm to construct such a triple.

The **main tool** for answering the above questions is the matrix $\mathcal{H}$ of Markov parameters:

$$\mathcal{H} := \begin{pmatrix} S_1 & S_2 & \cdots & S_k & S_{k+1} & \cdots \\ S_2 & S_3 & \cdots & S_{k+1} & S_{k+2} & \cdots \\ \\ \vdots & \vdots & & \vdots & \vdots & \\ \\ S_k & S_{k+1} & \cdots & S_{2k-1} & S_{2k} & \cdots \\ S_{k+1} & S_{k+2} & \cdots & S_{2k} & S_{2k+1} & \cdots \\ \\ \vdots & \vdots & & \vdots & \vdots & \end{pmatrix} \tag{3.40}$$

This is the **Hankel matrix**; it has infinitely many rows, infinitely many columns, and block Hankel structure, i.e. $(\mathcal{H})_{i,j} = S_{i+j-1}$, for $i, j > 0$. We start by listing conditions related to the realization problem.

Lemma 3.2 *Each statement below implies the one which follows:*
(a) *The sequence S_k, $k = 1, 2, \cdots$, is* **realizable**.
(b) *The formal power series $\sum_{k>0} S_k\xi^{-k}$ is* **rational**.
(c) *The sequence S_k, $k = 1, 2, \cdots$, satisfies a* **recursion** *with constant coefficients, i.e. there exist a positive integer r and constants α_i, $0 \le i < r$, such that*

$$\alpha_0 S_k + \alpha_1 S_{k+1} + \alpha_2 S_{k+2} + \ \cdots \ + \alpha_{r-2}S_{r+k-2} + \alpha_{r-1}S_{r+k-1} + S_{r+k} = 0, \ k > 0 \tag{3.41}$$

(d) *The rank of $\mathcal{H}$ is* **finite**.

Proof. **(a)** $\Rightarrow$ **(b)** Realizability implies (3.39). Hence

$$\sum_{k>0} S_k\xi^{-k} = \sum_{k>0} CA^{k-1}B\xi^{-k} = C\left(\sum_{k>0} A^{k-1}\xi^{-k}\right)B = C(\xi I - A)^{-1}B$$

This proves **(b)**.

(b) $\Rightarrow$ **(c)** Let $\det(\xi I - A) =: \alpha_0 + \alpha_1\xi + \ \cdots \ + \alpha_{r-1}\xi^{r-1} + \xi^r =: \chi_A(\xi)$. The previous relationship implies

$$\chi_A(\xi)\left(\sum_{k>0} S_k\xi^{-k}\right) = C\left[\text{adj}\,(\xi I - A)\right]B$$

where $\text{adj}\,(M)$ denotes the *adjoint* of the matrix M. On the left-hand side there are terms having both positive and negative powers of ξ, while on the right-hand side there are only terms having positive powers of ξ. Hence the coefficients of the negative powers of ξ on the left-hand side must be identically zero; this implies precisely (3.41).

(c) $\Rightarrow$ **(d)** Relationships (3.41) imply that the $(r+1)$-st block column of $\mathcal{H}$ is a linear combination of the previous r block columns. Furthermore, because of the block Hankel structure, every block column of $\mathcal{H}$ is a sub-column of the previous one; this implies that all block columns after the r-th are linearly dependent on the first r, which in turn implies the finiteness of the rank of $\mathcal{H}$. ∎

The following lemma describes a fundamental property of $\mathcal{H}$; it also provides a direct proof of the implication **(a)** $\Rightarrow$ **(d)**.

Lemma 3.3 **Factorization of** $\mathcal{H}$. *If the sequence of Markov parameters is* **realizable** *by means of the triple (C, A, B), $\mathcal{H}$ can be factored as follows:*

$$\mathcal{H} = \mathcal{O}(C, A)\mathcal{R}(A, B)$$

Consequently, if the sequence of Markov parameters is realizable the rank of $\mathcal{H}$ is finite.

Proof. If S_n, $n > 0$, is realizable the relationships $S_n = CA^{n-1}B$, $n > 0$, hold true. Hence:

$$\mathcal{H} = \begin{pmatrix} CB & CAB & \cdots \\ CAB & CA^2B & \cdots \\ \vdots & \vdots & \\ & & \end{pmatrix} = \mathcal{O}(C,A)\mathcal{R}(A,B)$$

It follows that: $\text{rank}\,\mathcal{H} \leq \max\{\text{rank}\,\mathcal{O}, \text{rank}\,\mathcal{R}\} \leq \text{size}\,(A)$. ■

In order to discuss the uniqueness issue of realizations, we need to recall the concept of equivalent systems defined by (3.23). In particular, proposition 3.1 asserts that equivalent triples have the same Markov parameters. Hence the best one can hope for in connection with the uniqueness question is that realizations be equivalent. Indeed as shown in the next section this holds for realizations with the smallest possible dimension.

3.3.1 The solution of the realization problem

We are now ready to answer the three questions posed at the beginning of the previous sub-subsection. This also proves the implication **(d)** $\Rightarrow$ **(a)**, and hence the equivalence of the statemenrs of lemma 3.2.

Theorem 3.5 Main Result.
(1) *The sequence S_k, $k > 0$, is realizable if, and only if,* $\text{rank}\,\mathcal{H} =: n < \infty$.
(2) *The state space dimension of any solution is at least n. All realizations which are minimal are both reachable and observable. Conversely, every realization which is reachable and observable is minimal.*
(3) *All minimal realizations are equivalent.*

Lemma 3.3 proves part **(1)** of the main theorem in one direction. To prove **(1)** in the other direction we will actually construct a realization assuming that the rank of $\mathcal{H}$ is finite.

Lemma 3.4 Silverman Realization Algorithm
Let $\text{rank}\,\mathcal{H} = n$. *Find an $n \times n$ submatrix Φ of $\mathcal{H}$ which has full rank. Construct the following matrices:*
(i) $\sigma\Phi \in \mathbb{R}^{n\times n}$*; it is composed of the same rows as Φ; its columns are obtained by shifting those of Φ by one block column (i.e. m columns).*
(ii) $\Gamma \in \mathbb{R}^{n\times m}$ *is composed of the same rows as Φ; its columns are the first m columns of $\mathcal{H}$.*
(iii) $\Lambda \in \mathbb{R}^{p\times n}$ *is composed of the same columns as Φ; its rows are the first p rows of $\mathcal{H}$.*
The triple (C, A, B), where $C := \Lambda$, $A := \Phi^{-1}\sigma\Phi$, and $B := \Phi^{-1}\Gamma$, is a realization of dimension n of the given sequence of Markov parameters.

Proof. By assumption there exist $n := \text{rank}\mathcal{H}$, columns of $\mathcal{H}$ which span its column space. Denote these columns by Φ_∞; note that the columns making up Φ_∞ need not be consecutive columns of $\mathcal{H}$. Let $\sigma\Phi_\infty$ denote the n columns of $\mathcal{H}$ obtained by shifting those of Φ_∞ by one block column, i.e. by m individual columns; let Γ_∞ denote the first m columns of $\mathcal{H}$. There exist *unique* matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, such that:

$$\sigma\Phi_\infty = \Phi_\infty A \tag{3.42}$$
$$\Gamma_\infty = \Phi_\infty B \tag{3.43}$$

Finally, define as C the first block row, i.e. the first p individual rows, of Φ_∞:

$$C := (\Phi_\infty)_1 \tag{3.44}$$

For this proof $(M)_k$, $k > 0$, will denote the k^{th} block row of the matrix M. Recall that the first block element of Γ_∞ is S_1, i.e. using our notation $(\Gamma_\infty)_1 = S_1$. Thus from (3.43), together with (3.44) follows

$$S_1 = (\Gamma_\infty)_1 = (\Phi_\infty B)_1 = (\Phi_\infty)_1 B = CB$$

For the next Markov parameter notice that $S_2 = (\sigma\Gamma_\infty)_1 = (\Gamma_\infty)_2$. Thus making use of (3.42)

$$S_2 = (\sigma\Gamma_\infty)_1 = (\sigma\Phi_\infty B)_1 = (\Phi_\infty AB)_1 = (\Phi_\infty)_1 AB = CAB$$

For the k^{th} Markov parameter, combining (3.43), (3.42) and (3.44), we obtain

$$S_k = (\sigma^{k-1}\Gamma_\infty)_1 = (\sigma^{k-1}\Phi_\infty B)_1 = (\Phi_\infty A^{k-1}B)_1 = (\Phi_\infty)_1 A^{k-1}B = CA^{k-1}B$$

Thus (C, A, B) is indeed a realization of dimension n. ■

The state dimension of a realization cannot be less than n; indeed, if such a realization existed, the rank of $\mathcal{H}$ would be less than n, which is a contradiction to the assumption that the rank of $\mathcal{H}$ is equal to n. Thus a realization of the sequence (S_k) of dimension equal to $\operatorname{rank}\mathcal{H}$ is called *minimal realization*; notice that the Silverman algorithm constructs minimal realizations. In this context the following holds true.

Lemma 3.5 *A realization of the sequence (S_k) is minimal if, and only if, it is reachable and observable.*

Proof. Let (C, A, B) be some realization of S_n, $n > 0$. Since $\mathcal{H} = \mathcal{O}\mathcal{R}$:

$$\operatorname{rank}\mathcal{H} \leq \min\{\operatorname{rank}\mathcal{O}, \operatorname{rank}\mathcal{R}\} \leq \operatorname{size}(A)$$

Let $(\hat{C}, \hat{A}, \hat{B})$ be a reachable and observable realization. Since $\mathcal{H} = \hat{\mathcal{O}}\hat{\mathcal{R}}$, and each of the matrices $\hat{\mathcal{O}}$, $\hat{\mathcal{R}}$ contain a nonsingular matrix of size equal to the size of $\hat{A}$, we conclude that $\operatorname{size}(\hat{A}) \leq \operatorname{rank}\mathcal{H}$. This concludes the proof. ■

We are now left with the proof of part **(3)** of the main theorem, namely that minimal realizations are equivalent. We will only provide the proof for a special case; the proof of the general case follows along similar lines.

Outline of Proof. Single-input, single-output case (i.e. $p = m = 1$). Let (C_i, A_i, B_i), $i = 1, 2$, be minimal realizations of $\mathcal{S}$. We will show the existence of a transformation T, $\det T \neq 0$, such that (3.23) holds. From lemma 3.3 we conclude that

$$\mathcal{H}_{n,n} = \mathcal{O}_n^1\mathcal{R}_n^1 = \mathcal{O}_n^2\mathcal{R}_n^2 \tag{3.45}$$

where the superscript is used to distinguish between the two different realizations. Furthermore, the same lemma also implies $\mathcal{H}_{n,n+1} = \mathcal{O}_n^1[B_1 \;\; A_1\mathcal{R}_n^1] = \mathcal{O}_n^2[B_2 \;\; A_2\mathcal{R}_n^2]$, which in turn yields

$$\mathcal{O}_n^1 A_1 \mathcal{R}_n^1 = \mathcal{O}_n^2 A_2 \mathcal{R}_n^2 \tag{3.46}$$

Because of minimality, the following determinants are non-zero: $\det\mathcal{O}_n^i \neq 0$, $\det\mathcal{R}_n^i \neq 0$, $i = 1, 2$. We now define $T = (\mathcal{O}_n^1)^{-1}\mathcal{O}_n^2 = \mathcal{R}_n^1(\mathcal{R}_n^2)^{-1}$. Equation (3.45) implies $C_1 = C_2T^{-1}$ and $B_1 = TB_2$, while (3.46) implies $A_1 = TA_2T^{-1}$. ■

3.3.2 Realization of proper rational matrix functions

Given is a $p \times m$ matrix $H(\xi)$ with proper rational entries, i.e. entries whose numerator degree is no larger than the denominator degree. Consider first the scalar case, i.e. $p = m = 1$. We can write

$$H(\xi) = D + \frac{p(\xi)}{q(\xi)}$$

where D is a constant in $\mathbb{R}$ and p, q are polynomials in ξ

$$p(\xi) = p_0 + p_1\xi + \cdots + p_{\nu-1}\xi^{\nu-1}, \; q(\xi) = q_0 + q_1\xi + \cdots + q_{\nu-1}\xi^{\nu-1} + \xi^\nu, \; p_i, q_i \in \mathbb{R}$$

In terms of these coefficients p_i and q_i we can write down a realization of $H(\xi)$ as follows

$$\boldsymbol{\Sigma}_H := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) := \left(\begin{array}{ccccc|c} 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ -q_0 & -q_1 & -q_2 & \cdots & -q_{\nu-1} & 1 \\ \hline p_0 & p_1 & p_2 & \cdots & p_{\nu-1} & D \end{array}\right) \in \mathbb{R}^{(\nu+1)\times(\nu+1)} \tag{3.47}$$

It can be shown that $\boldsymbol{\Sigma}_H$ is indeed a realization of H, i.e

$$H(\xi) = D + C(\xi I_\nu - A)^{-1}B$$

This realization is reachable but not necessarily observable; this means that the rank of the associated Hankel matrix is at most ν. The realization is in addition, observable if the polynomials p and q are coprime. Thus (3.47) is *minimal* if p, q are coprime. In this case the rank of the associated Hankel matrix $\mathcal{H}$ is precisely ν.

In the general case, we can write

$$H(\xi) = D + \frac{1}{q(\xi)}P(\xi)$$

where q is a scalar polynomial which is the least common multiple of the denominators of the entries of H, and P is a polynomial matrix of size $p \times m$:

$$P(\xi) = P_0 + P_1\xi + \cdots + P_{\nu-1}\xi^{\nu-1},\ P_i \in \mathbb{R}^{p\times m},\ q(\xi) = q_0 + q_1\xi + \cdots + q_{\nu-1}\xi^{\nu-1} + \xi^\nu,\ q_i \in \mathbb{R}$$

The construction given above provides a realization:

$$\boldsymbol{\Sigma}_H := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) := \left(\begin{array}{ccccc|c} 0_m & I_m & 0_m & & 0_m & 0_m \\ 0_m & 0_m & I_m & & 0_m & 0_m \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_m & 0_m & 0_m & & I_m & 0_m \\ -q_0 I_m & -q_1 I_m & -q_2 I_m & \cdots & -q_{\nu-1}I_m & I_m \\ \hline P_0 & P_1 & P_2 & \cdots & P_{\nu-1} & D \end{array}\right) \in \mathbb{R}^{(\nu m+p)\times(\nu m+m)} \tag{3.48}$$

where 0_m is a square zero matrix of size m, and I_m is the identity matrix of the same size. Unlike the scalar case however, the realization $\boldsymbol{\Sigma}_H$ need not be minimal. One way to obtain a minimal realization is by applying the Silverman algorithm; in this case H has to be expanded into a formal power series

$$H(\xi) = S_0 + S_1\xi^{-1} + S_2\xi^{-2} + \ \cdots \ + S_t\xi^{-t} + \ \cdots$$

The Markov parameters are computed using the following relationship. Given the polynomial q as above, let

$$q^{(k)}(\xi) := \xi^{\nu-k} + q_{n-1}\xi^{\nu-k-1} + \ \cdots \ + + q_{k+1}\xi + q_k,\ k = 1, \cdots, \nu$$

denote its ν *pseudo-derivative* polynomials. It follows that the numerator polynomial $P(\xi)$ is related with the Markov parameters S_k and the denominator polynomial q, as follows:

$$P(\xi) = S_1 q^{(1)}(\xi) + S_2 q^{(2)} + \ \cdots \ + S_{\nu-1}q^{(\nu-1)} + S_\nu q^{(\nu)}$$

This can be verified by direct calculation. Alternatively, assume that $H(\xi) = C(\xi I - A)^{-1}B$, and let $q(\xi)$ denote the characteristic polynomial of A. Then

$$\operatorname{adj}(\xi I - A) = q^{(\nu)}(\xi)A^{\nu-1} + q^{(\nu-1)}(\xi)A^{\nu-2} + \ \cdots \ q^{(2)}(\xi)A^1 + q^{(1)}(\xi)I$$

The result on P follows by noting that $P(\xi) = C\mathrm{adj}\,(\xi I - A)B$.

Since H is rational, the rank of the ensuing *Hankel* matrix associated with the sequence of Markov parameters S_n, $n > 0$, is guaranteed to have *finite rank*. In particular the following upper bound holds

$$\mathrm{rank}\mathcal{H} \leq \min\{\nu m, \nu p\} \tag{3.49}$$

A concept often used is that of the *McMillan degree* of a rational matrix function. For proper rational matrix functions H the McMillan degree turns out to be equal to the rank of the associated Hankel matrix $\mathcal{H}$; in other words the McMillan degree in this case is equal to the dimension of any minimal realization of H.

Example 3.5 The first three sequences listed in example 3.4 are *realizable*, while the last two are not. We will now investigate in detail the realization problem for the third sequence, the *Fibonacci numbers*. It is constructed according to the rule $S_{k+2} = S_{k+1} + S_k$, $k > 0$, with $S_1 = 1$, $S_2 = 2$. The associated Hankel matrix (3.40) is

$$\mathcal{H} := \begin{pmatrix} 1 & 2 & 3 & 5 & 8 & 13 & \cdots \\ 2 & 3 & 5 & 8 & 13 & 21 & \cdots \\ 3 & 5 & 8 & 13 & 21 & 34 & \cdots \\ 5 & 8 & 13 & 21 & 34 & 55 & \cdots \\ 8 & 13 & 21 & 34 & 55 & 89 & \cdots \\ 13 & 21 & 34 & 55 & 89 & 144 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It readily follows from the law of construction of the sequence, that the rank of the Hankel matrix is two. Φ will be chosen so that it contains rows 2, 4 and columns 2, 5 of $\mathcal{H}$:

$$\Phi = \begin{pmatrix} 3 & 13 \\ 8 & 34 \end{pmatrix},\ \sigma\Phi = \begin{pmatrix} 5 & 21 \\ 13 & 55 \end{pmatrix},\ \Gamma = \begin{pmatrix} 2 \\ 5 \end{pmatrix},\ \Lambda = (2\ \ 8)$$

It follows that

$$A = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix},\ B = \frac{1}{2}\begin{pmatrix} -3 \\ 1 \end{pmatrix},\ C = (2\ \ 8)$$

Furthermore we have $\sum_{k>0} S_k \xi^{-k} = \frac{\xi+1}{\xi^2-\xi-1}$. ■

Example 3.6 Realization of $L_k(s) = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s+2} \\ 1 & \frac{k}{s+3} \end{bmatrix}$, $k \in \mathbb{R}$, (see example 3.1).

For $k = 0$ we have the following realization:

$$A_0 = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix},\ B_0 = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix},\ C_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ D_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

It is readily checked that it is minimal. For $k \neq 0$, we have the following Markov parameters:

$$S_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ S_t = \begin{pmatrix} 1 & (-2)^{t-1} \\ 0 & (-3)^{t-1}k \end{pmatrix},\ t = 1, 2, \cdots$$

The corresponding block Hankel matrix is

$$\mathcal{H} = \left[\begin{array}{c|c|c|c} S_1 & S_2 & S_3 & \cdots \\ \hline S_2 & S_3 & S_4 & \cdots \\ \hline S_3 & S_4 & S_5 & \cdots \\ \hline \vdots & \vdots & \vdots & \ddots \end{array}\right] = \left[\begin{array}{cc|cc|cc|c} 1 & 1 & 1 & -2 & 1 & 4 & \cdots \\ 0 & k & 0 & -3k & 0 & 9k & \cdots \\ \hline 1 & -2 & 1 & 4 & 1 & -8 & \cdots \\ 0 & -3k & 0 & 9k & 0 & -27k & \cdots \\ \hline 1 & 4 & 1 & -8 & 1 & 16 & \cdots \\ 0 & 9k & 0 & -27k & 0 & 81k & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right], \; k \neq 0.$$

According to (3.49), the rank of $\mathcal{H}$ is less than 6. By considering the finite submatrix of $\mathcal{H}$ composed of the first three block columns and block rows, it turns out that the rank of $\mathcal{H}$ is 3. In fact the matrix Φ composed of rows $\{1, 2, 3\}$ and columns $\{1, 2, 4\}$ is non-singular. Thus

$$\Phi = \begin{bmatrix} 1 & 1 & -2 \\ 0 & k & -3\,k \\ 1 & -2 & 4 \end{bmatrix}, \; \sigma\Phi = \begin{bmatrix} 1 & -2 & 4 \\ 0 & -3\,k & 9\,k \\ 1 & 4 & -8 \end{bmatrix}, \; \Gamma = \begin{bmatrix} 1 & 1 \\ 0 & k \\ 1 & -2 \end{bmatrix}, \; \Lambda = \begin{bmatrix} 1 & 1 & -2 \\ 0 & k & -3\,k \end{bmatrix}.$$

The algorithm of lemma 3.4 yields the following minimal realization

$$\left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right] = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -6 & 0 & 1 \\ 0 & 1 & -5 & 0 & 0 \\ \hline 1 & 1 & -2 & 0 & 0 \\ 0 & k & -3\,k & 0 & 0 \end{array}\right], \; k \neq 0.$$

■

4 Time and frequency domain interpretation of various norms

Recall section 3.1 and in particular the definition of the convolution operator $\mathcal{S}$ (3.4), (3.7); first a new operator will be defined, which is obtained by restricting the domain and the codomain of $\mathcal{S}$. This is the Hankel operator $\mathcal{H}$. The significance of this operator lies in the fact that in contrast to $\mathcal{S}$, it has a *discrete* set of *singular values*.

Next we will compute various norms as well as the spectra of $\mathcal{S}$ and $\mathcal{H}$. The calculations will be performed for the continuous-time case, the results for the discrete-time case following similarly. This section is concluded with an account as to where and how these norms are used in control systems design.

4.1 The convolution operator and the Hankel operator

Given a linear, time invariant, not necessarily causal system $\mathbf{\Sigma}$, its convolution operator $\mathcal{S}$ induces an operator which is of interest in the theory of linear systems: the **Hankel operator** $\mathcal{H}$ which is defined as follows:

$$\mathcal{H} : \ell^m(\mathbb{Z}_-) \longrightarrow \ell^p(\mathbb{Z}_+), \; u_- \longmapsto y_+ \; := \; \mathcal{H}(u_-) \; \text{ where } \; y_+(t) := \sum_{k=-\infty}^{-1} S_{t-k} u_-(k), \; t \geq 0 \qquad (4.1)$$

Thus the Hankel operator $\mathcal{H}$ is obtained from the convolution operator $\mathcal{S}$ by restricting its domain and codomain. The matrix representations of $\mathcal{H}$, is given by the lower-left block of the matrix representation of $\mathcal{S}$; rearranging the entries of u_- we obtain

$$\underbrace{\begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \end{pmatrix}}_{y_+} = \underbrace{\begin{pmatrix} S_1 & S_2 & S_3 & \cdots \\ S_2 & S_3 & S_4 & \cdots \\ S_3 & S_4 & S_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{\mathcal{H}} \underbrace{\begin{pmatrix} u(-1) \\ u(-2) \\ u(-3) \\ \vdots \end{pmatrix}}_{u_-}$$

The ℓ_2-induced norm of $\mathcal{H}$ is by definition its largest singular value

$$\| \mathcal{H} \|_{\ell_2-\mathrm{ind}} = \sigma_{max}(\mathcal{H}) = \| \mathbf{\Sigma} \|_H$$

The quantity $\| \mathbf{\Sigma} \|_H$ is called the **Hankel norm** of the system described by the convolution operator $\mathcal{S}$.

If in addition the system is stable, by combining the discrete-time versions of theorems 2.2 and 2.3, it follows that the ℓ_2-induced norm of h is equal to the h_∞-Schatten norm of its transform H:

$$\| h \|_{\ell_2-\mathrm{ind}} = \| H \|_{h_\infty}$$

For short, this relationship is written $\| h \|_2 = \| H \|_\infty$; we often refer to this quantity as the **h_∞-norm of $\mathbf{\Sigma}$**.

Given a linear, time invariant, continuous-time not necessarily causal system, similarly to the discrete-time case, the convolution operator $\mathcal{S}$ induces a **Hankel operator** $\mathcal{H}$, which is defined as follows:

$$\mathcal{H} : \mathcal{L}^m(\mathbb{R}_-) \longrightarrow \mathcal{L}^p(\mathbb{R}_+),\ u_- \longmapsto y_+ := \mathcal{H}(u_-) \ \text{ where } \ y_+(t) := \int_{-\infty}^0 h(t-\tau)u_-(\tau)d\tau,\ t \geq 0 \tag{4.2}$$

The $\mathcal{L}_2$-induced norm of $\mathcal{H}$ is

$$\| \mathcal{H} \|_{\mathcal{L}_2-\mathrm{ind}} = \sigma_{max}(\mathcal{H}) =: \| \mathbf{\Sigma} \|_H \tag{4.3}$$

The quantity $\| \mathbf{\Sigma} \|_H$ is called the **Hankel norm** of the system described by the convolution operator $\mathcal{S}$.

As in the discrete-time case, if the system is stable, by combining theorems 2.2 and 2.3 follows that the $\mathcal{L}_2$-induced norm of the system defined by the kernel h is equal to the $\mathcal{H}_\infty$-Schatten norm of its transform H:

$$\| h \|_{\mathcal{L}_2-\mathrm{ind}} = \| H \|_{\mathcal{H}_\infty}$$

For short, this relationship is written $\| h \|_2 = \| H \|_\infty$, and we refer to this quantity as the **$\mathcal{H}_\infty$-norm of $\mathbf{\Sigma}$**.

Remark 4.1 Significance of $\mathcal{H}$. As we will see in the two sections that follow, the Hankel operator has a discrete set of singular values. In contrast, this is not the case with the convolution operator. Therefore the singular values of $\mathcal{H}$ play a fundamental role in control design and system approximation. ■

4.2 Computation of the singular values of $\mathcal{S}$

We will now compute the various norms assuming that the linear system is continuous-time and is given in state space form: $\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$; it follows from (3.20) that its impulse response is $h(t) = Ce^{At}B + \delta(t)Du$, $t \geq 0$. Analogous results hold for discrete-time systems.

First, we will compute the *adjoint* $\mathcal{S}^*$ of the operator $\mathcal{S}$. By definition, given the inner product $\langle \cdot, \cdot \rangle$, $\mathcal{S}^*$ is the unique operator satisfying

$$\langle y, \mathcal{S}u \rangle = \langle \mathcal{S}^* y, u \rangle$$

for all y, u in the appropriate spaces.

By (3.8), $\mathcal{S}$ is an integral operator with kernel $h(\cdot)$; a relatively straightforward calculation using the above definition, shows that $\mathcal{S}^*$ is also an integral operator with *kernel* h^* and with time running backwards:

$$\mathcal{S}^* : \mathcal{L}^p(\mathbb{R}) \longrightarrow \mathcal{L}^m(\mathbb{R}) y \longmapsto u := \mathcal{S}^*(y) \text{ where } u(t) = \int_{-\infty}^{\infty} h^*(-t+\tau)y(\tau)d\tau,\ t \in \mathbb{R}$$

Consider the case of square systems $m = p$. Let $u_0(t) = v_0 e^{j\omega_0 t}$, $v_0 \in \mathbb{R}^m$, $t \in \mathbb{R}$, be a periodic input. It readily follows that

$$\mathcal{S}(u_0)(t) = H(j\omega_0)v_0 e^{j\omega_0 t} \tag{4.4}$$

and

$$\mathcal{S}^*(\mathcal{S}(u_0))(t) = H^*(-j\omega_0)H(j\omega_0)v_0 e^{j\omega_0 t} \tag{4.5}$$

(4.4) shows that u_0 is an eigenfunction of $\mathcal{S}$ provided that v_0 is an eigenvector of $H(j\omega_0)$. Furthermore (4.5) shows that u_0 is a right singular function of $\mathcal{S}$ provided that v_0 is a right singular vector of the hermitian matrix $H^*(-j\omega_0)H(j\omega_0)$. We conclude that the spectrum of $\mathcal{S}$ is composed of the eigenvalues of $H(j\omega)$ for all $\omega \in \mathbb{R}$, while the spectrum of $\mathcal{S}^*\mathcal{S}$ is every point in the interval

$$(\underline{\sigma}, \overline{\sigma}) : \ \underline{\sigma} := \inf_{\omega} \sigma_{min}\left(H^*(-j\omega)H(j\omega)\right), \ \ \overline{\sigma} := \sup_{\omega} \sigma_{max}\left(H^*(-j\omega)H(j\omega)\right)$$

Notice that the eigen-functions and singular-functions of $\mathcal{S}$ are signals of finite *power* (root-mean-square energy) but not finite *energy*.

Example 4.1 We will illustrate the preceeding discussion by computing the singular values of the convolution operator $\mathcal{S}$ associated with the discrete-time system: $y(k+1) - ay(k) = bu(k)$. The impulse response of this system is: $S_0 = 0$, $S_k = ba^{k-1}$, $k > 0$. Hence from (3.6) follows:

$$\mathcal{S} = b \left(\begin{array}{ccc|ccc} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & 0 & 0 & 0 & \cdots \\ \hline \cdots & a & 1 & 0 & 0 & \cdots \\ \cdots & a^2 & a & 1 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

To compute the singular values of $\mathcal{S}$, we will first consider the finite Toeplitz submatrices $\mathcal{S}_n$ of $\mathcal{S}$:

$$\mathcal{S}_n = b \begin{pmatrix} 1 & & & & \\ a & 1 & & & \\ \vdots & a & & & \\ & \vdots & \ddots & & \\ a^{n-2} & & & 1 & \\ a^{n-1} & a^{n-2} & \cdots & a & 1 \end{pmatrix} \in \mathbb{R}^{n \times n}$$

It follows that the singular values of $\mathcal{S}_n$ lie between

$$\frac{|b|}{1+|a|} \leq \sigma_i(\mathcal{S}_n^*\mathcal{S}_n) \leq \frac{|b|}{1-|a|},\ i = 1, 2, \ \cdots, n$$

This relationship is valid for all n. Hence in the limit $n \to \infty$, the singular values of $\mathcal{S}$ are composed of the interval $\left[\frac{|b|}{1+|a|}, \frac{|b|}{1-|a|}\right]$. Let us now compare the singular values of $\mathcal{S}$ with the singular values of $\mathcal{S}_{100}$, for $a = \frac{1}{2}$ and $b = 3$. The largest, smallest singular values of this matrix are: $\sigma_1(\mathcal{S}_n) = 5.9944$, $\sigma_{100}(\mathcal{S}_n) = 2.002$, respectively. The corresponding singular values of $\mathcal{S}$ are 6, 2 respectively. ■

4.3 Computation of the singular values of $\mathcal{H}$

We begin with a definition.

Definition 4.1 *The* **Hankel singular values** *of the stable system* $\mathbf{\Sigma}$*, denoted by:*

$$\sigma_1(\mathbf{\Sigma}) > \cdots > \sigma_q(\mathbf{\Sigma}) \text{ with multiplicity } r_i,\ i = 1, \cdots, q,\ \sum_{i=1}^{q} r_i = n \tag{4.6}$$

are the singular values of $\mathcal{H}_{\mathbf{\Sigma}}$ *defined by (4.1), (4.2). The* **Hankel norm** *of* $\mathbf{\Sigma}$ *is the largest Hankel singular value:*

$$\|\mathbf{\Sigma}\|_H := \sigma_1(\mathbf{\Sigma})$$

The Hankel operator of a not necessarily stable system $\mathbf{\Sigma}$*, is defined as the Hankel operator of its stable and causal part* $\mathbf{\Sigma}_+$*:* $\mathcal{H}_{\mathbf{\Sigma}} := \mathcal{H}_{\mathbf{\Sigma}_+}$.

In order to compute the singular values of $\mathcal{H}$ we need its *adjoint* $\mathcal{H}^*$. Recall (4.2). For continuous-time systems the adjoint is defined as follows:

$$\mathcal{H}^* : \mathcal{L}^p(\mathbb{R}_+) \longrightarrow \mathcal{L}^m(\mathbb{R}_-),\ y_+ \longmapsto u_- := \mathcal{H}^*(y_+) \text{ where } u_-(t) := \int_0^\infty h^*(-t+\tau)y(\tau)d\tau,\ t \leq 0$$

In the sequel we will assume that the underlying system is finite dimensional, i.e. the rank of the Hankel matrix derived from the corresponding Markov parameters S_t of h is finite. Consequently, by section 3.3, there exist a triple (C, A, B) such that (3.20) holds:

$$h(t) = Ce^{At}B,\ t > 0$$

It follows that for a given u

$$(\mathcal{H}u)(t) = \int_{-\infty}^{0} h(t-\tau)u(\tau)d\tau = Ce^{At} \underbrace{\int_{-\infty}^{0} e^{-A\tau}Bu(\tau)d\tau}_{=:x_i} = Ce^{At}x_i,\ x_i \in \mathbb{R}^n$$

Moreover

$$(\mathcal{H}^*\mathcal{H}u)(t) = B^*e^{A^*t}\left(\int_0^\infty e^{A^*\sigma}C^*Ce^{A\sigma}d\sigma\right)x_i = B^*e^{A^*t}\mathcal{Q}x_i$$

where the expression in parenthesis is $\mathcal{Q}$, the infinite observability gramian defined by (3.32). The requirement for u to be an eigenfunction of $\mathcal{H}^*\mathcal{H}$, is that this last expression be equal to $\sigma_i^2 u(t)$. This implies:

$$u(t) = \frac{1}{\sigma_i^2}B^*e^{-A^*t}\mathcal{Q}x_i,\ t \leq 0$$

Substituting u in the expression for x_i we obtain

$$\frac{1}{\sigma_i^2}\left(\int_{-\infty}^{0} e^{-A\tau}BB^*e^{-A^*\tau}d\tau\right)\mathcal{Q}x_i = x_i$$

Recall the definition (3.31) of the infinite reachability gramian; this equation becomes $\mathcal{P}\mathcal{Q}x_i = \sigma_i^2 x_i$. We conclude that the (non-zero) singular values of the Hankel operator $\mathcal{H}$ are the eigenvalues of the product of the infinite gramians $\mathcal{P}$ and $\mathcal{Q}$ of the system. Therefore $\mathcal{H}$, in contrast to $\mathcal{S}$, has a discrete set of singular values. It can be shown that the above expression holds equally for discrete-time systems where $\mathcal{P}$, $\mathcal{Q}$ are the infinite gramians obtained by solving the *discrete Lyapunov* or *Stein* equations (3.37) respectively. In summary we have:

Lemma 4.1 *Given the reachable, observable and stable discrete- or continuous-time system* $\mathbf{\Sigma}$ *of dimension* n*, the positive square roots of the eigenvalues of* $\mathcal{P}\mathcal{Q}$ *are the* **Hankel singular values** *of* $\mathbf{\Sigma}$*:*

$$\sigma_k(\mathbf{\Sigma}) = \sqrt{\lambda_k(\mathcal{P}\mathcal{Q})}\,,\ k = 1,\ \cdots\ ,\ n \tag{4.7}$$

Furthermore σ_1 *is the* **Hankel norm** *of* $\mathbf{\Sigma}$*.*

Recall the definition of equivalent systems (3.23). Under equivalence, the gramians are transformed as follows

$$\tilde{\mathcal{P}} = T\mathcal{P}T^*,\ \tilde{\mathcal{Q}} = T^{-*}\mathcal{Q}T^{-1} \ \Rightarrow\ \tilde{\mathcal{P}}\tilde{\mathcal{Q}} = T\,(\mathcal{P}\mathcal{Q})\,T^{-1}$$

Therefore, the product of the two gramians of equivalent systems is related by similarity transformation, and hence has the **same eigenvalues**.

Corollary 4.1 *The Hankel singular values are input-output invariants of* $\mathbf{\Sigma}$*.*

Remark 4.2 (a) For *discrete-time systems*: $\sigma_k(\mathbf{\Sigma}) = \sigma_k(\mathcal{H})$, i.e. the singular values of the system are the singular values of the (block) Hankel matrix defined by (3.40). For *continuous-time systems* however $\sigma_k(\mathbf{\Sigma})$ are the singular values of a continuous-time *Hankel operator*. They are **not** equal to the singular values of the associated matrix of Markov parameters.

(b) It should be noticed that following proposition 3.5 the Hankel singular values of a continuous-time stable system and those of a discrete-time stable system related by means of the bilinear transformation $s = \frac{z-1}{z+1}$ are the *same*. ■

4.4 Computation of various norms

4.4.1 The $\mathcal{H}_2$ norm

This norm is defined as the $\mathcal{L}_2$ norm of the impulse response (in the time domain):

$$\parallel \mathbf{\Sigma} \parallel_{\mathcal{H}_2} := \parallel h(t) \parallel_{\mathcal{L}_2(\mathbb{R}_+)} = \parallel H(s) \parallel_{\mathcal{H}_2(\mathbb{C}_+)} \tag{4.8}$$

Therefore it exists only if $D = 0$ and $\mathbf{\Sigma}$ is stable, i.e. the eigenvalues of A are in the left-half of the complex plane; in this case there holds (2.17):

$$\parallel \mathbf{\Sigma} \parallel^2_{\mathcal{H}_2} = \int_0^\infty \mathrm{trace}\,[h^*(t)h(t)]dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}\,[H^*(-j\omega)H(j\omega)]d\omega$$

where the second equality is a consequence of Parseval's theorem. Thus using (3.32) we obtain

$$\parallel \mathbf{\Sigma} \parallel^2_{\mathcal{H}_2} = \int_0^\infty \mathrm{trace}\,[B^* e^{A^*t}C^*Ce^{At}B]dt = \mathrm{trace}\,[B^*\mathcal{Q}B^*]$$

Furthermore since $\mathrm{trace}\,[h^*(t)h(t)] = \mathrm{trace}\,[h(t)h^*(t)]$, using (3.31), this last expression is also equal to $C\mathcal{P}C^*$; therefore

$$\parallel \mathbf{\Sigma} \parallel_{\mathcal{H}_2} = \sqrt{\mathrm{trace}\,[B^*\mathcal{Q}B]} = \sqrt{\mathrm{trace}\,[C\mathcal{P}C^*]} \tag{4.9}$$

An interesting question is whether the $\mathcal{H}_2$ norm is *induced*. It can be shown that the 2-∞ induced norm of the convolution operator is:

$$\|\mathcal{S}\|_{2,\infty} = \sup_{u\neq 0}\frac{\|y\|_\infty}{\|u\|_2} = \sqrt{\lambda_{\max}(C\mathcal{P}C^*)}\ \text{ where }\ \|y\|_\infty = \sup_t\ \max_i\ y_i(t)$$

Consequently, in the single-input single-output ($m = p = 1$) case, the $\mathcal{H}_2$ norm is an induced norm. This norm can be interpreted as the maximum amplitude of the output which results from finite energy input signals.

4.4.2 The $\mathcal{H}_\infty$ norm

According to (2.18), if $\mathbf{\Sigma}$ is stable, i.e. the eigenvalues of A have negative real parts,

$$\|\mathbf{\Sigma}\|_{\mathcal{H}_\infty} = \sup_\omega \sigma_{max}(H(j\omega))$$

Consider the rational function

$$K(j\omega) = \gamma^2 I_m - H^*(-j\omega)H(j\omega)$$

If γ is bigger than the $\mathcal{H}_\infty$ norm od $\mathbf{\Sigma}$, there is no real ω, such that $K(j\omega)$ is zero. Thus if we define $K(s) = \gamma^2 - H^*(-s)H(s)$, where $H^*(-s) = B^*(-sI - A^*)^{-1}C^* + D^*$ is the adjoint system, the $\mathcal{H}_\infty$ norm of $\mathbf{\Sigma}$ is less than γ if, and only if, $K^{-1}(s)$ has no pure imaginary poles. The A matrix of K^{-1} is

$$A_K(\gamma) = \begin{bmatrix} A & \gamma^{-1}BB^* \\ -\gamma^{-1}C^*C & -A^* \end{bmatrix}$$

Hence we have the result

Proposition 4.1 $\|\mathbf{\Sigma}\|_{\mathcal{H}_\infty} < \gamma$ *if, and only if, the matrix $A_K(\gamma)$ has no complex eigenvalues.*

For a sufficiently large $\bar{\gamma}$, $A_K(\gamma)$ has no pure imaginary eigenvalues, while for a sufficiently small $\underline{\gamma}$, it does. The algorithm used to find an approximation of the $\mathcal{H}_\infty$ norm consists in dissecting the interval $[\underline{\gamma}, \bar{\gamma}]$: let $\tilde{\gamma} = \frac{\underline{\gamma}+\bar{\gamma}}{2}$. If $F(\tilde{\gamma})$ has imaginary eigenvalues then the interval above is substituted by the interval where $\underline{\gamma} = \tilde{\gamma}$, otherwise by $\bar{\gamma} = \tilde{\gamma}$; both of these intervals have now half the lenght of the previous interval. The procedure continues until the difference $\bar{\gamma} - \underline{\gamma}$ is sufficiently small.

The condition that $A_K(\gamma)$ have no pure imaginary eigenvalues is equivalent to the condition that the Riccati equation $A^*X + XA + \gamma^{-1}XBB^*X + \gamma^{-1}C^*C = 0$, have a positive definite solution X.

4.4.3 The Hilbert-Schmidt norm

An operator $\mathcal{I} : X \to Y$, where X, Y are Hilbert spaces, is *Hilbert-Schmidt*, if there exists a complete orthonormal sequence $\{x_n\} \in X$, such that

$$\sum_{n>0} \|\mathcal{I}(x_n)\| < \infty$$

This property can be readily checked for integral operators $\mathcal{I}$. The integral operator

$$\mathcal{I}(w)(x) = \int_a^b k(x,y)w(y)dy,\ x \in [c,\ d],$$

is a *Hilbert-Schmidt* operator, if its kernel k is square intergrable in both variables:

$$\kappa^2 = \text{trace}\left[\int_a^b \int_c^d k^*(x,y)k(x,y)dxdy\right]$$

is finite. In this case κ is the Hilbert-Schmidt norm of $\mathcal{I}$. Such operators are compact and hence have a discrete spectrum, where each eigenvalue has finite multiplicity, and the only accumulation point is zero.

It readily follows that the convolution operator $\mathcal{S}$ associated with $\mathbf{\Sigma}$ is *not* Hilbert-Schmidt, while the Hankel operator $\mathcal{H}$ is. In particular:

$$\kappa^2 = \text{trace}\left[\int_0^\infty \int_{-\infty}^0 h^*(t-\tau)h(t-\tau)\ d\tau\ dt\right]$$

Assuming that the system is stable, this expression is equal to

$$\begin{aligned}
\kappa^2 &= \text{trace} \int_0^\infty \int_{-\infty}^0 B^* e^{A^*(t-\tau)} C^* C e^{A(t-\tau)} B \, d\tau \, dt \\
&= \text{trace} \int_0^\infty B^* e^{A^* t} \left[\int_{-\infty}^0 e^{-A^*\tau} C^* C e^{-At} dt \right] e^{At} B \, dt \\
&= \text{trace} \int_0^\infty B^* e^{A^* t} \, Q \, e^{At} B \, dt \\
&= \text{trace} \int_0^\infty e^{At} B B^* e^{A^* t} \, Q \, dt \; = \; \text{trace} \, [\mathcal{P}\mathcal{Q}] = \sigma_1^2 + \; \cdots \; + \sigma_n^2
\end{aligned}$$

where σ_i are the Hankel singular values of the system $\mathbf{\Sigma}$.

Example 4.2 For the discrete-time system $y(k+1) - ay(k) = bu(k)$, discussed earlier, the Hankel operator is:

$$\mathcal{H} = b \begin{pmatrix} 1 & a & a^2 & \cdots \\ a & a^2 & a^3 & \cdots \\ a^2 & a^3 & a^4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This operator has a single non-zero singular value, which turns out to be

$$\sigma_1(\mathcal{H}) = \frac{|b|}{1 - a^2}$$

In this case, since $\mathcal{H}$ is symmetric $\lambda_1(\mathcal{H}) = \pm\sigma_1(\mathcal{H})$. Furthermore, the Hilbert-Schmidt norm (which is equal to the trace of $\mathcal{H}$, up to a sign) is $\frac{b}{1-a^2}$. ■

Hankel singular values and the Nyquist diagram

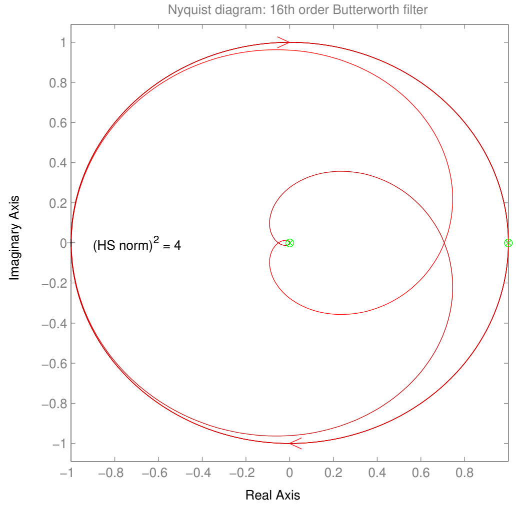
Consider the singular values $\sigma_i(\mathcal{H})$, where $\mathcal{H}$ is the Hankel operator of linear, finite-dimensional, system. In the SISO case the following result holds. It provides a *frequency domain* interpretation of the Frobenius or Hilbert-Schmidt norm of the Hankel operator:

$$\left. \begin{array}{c} \text{Area of Nyquist diagram} \\ \text{including multiplicities} \end{array} \right\} = \pi \cdot \underbrace{(\sigma_1^2 + \; \cdots \; + \sigma_n^2)}_{(\text{Hilbert}-\text{Schmidt Norm of } \mathcal{H})^2} = \pi \, \|\mathcal{H}\|_{\text{HS}}^2$$

Example 4.3 We consider a 16th order continuous-time Butterworth filter. In the left-hand side plot of figure 5, its Nyquist diagram is shown; in the middle are the 16 Hankel singular values; on the right-hand side is the impulse response of the system. The Nyquist diagram winds 3 times around the origin in almost perfect circles of radius 1; then it winds once more. The area of this diagram, multiplicities included is exactly 4π; it can be verified that the sum of the 16 Hankel singular values is exactly 4. Furthermore we compute the following norms:

$$\|\mathbf{\Sigma}\|_{2,\infty} = .5464, \;\; \|\mathbf{\Sigma}\|_H = .9996, \;\; \|\mathbf{\Sigma}\|_{\mathcal{H}_\infty} = 1, \;\; \|\mathbf{\Sigma}\|_{\infty,\infty} = 2.1314, \;\; 2\sum_i \sigma_i = 9.5197$$

Following the relationships in the table below we have $.9996 < 1$, which means that the largest Hankel singular value is very close to 1, but still, smaller. Also $.5646 < 1 < 2.1314 < 9.5197$, which means that the $2, \infty$ induced norm which is equal to the $\mathcal{H}_2$ norm, is less that the $2, 2$-induced norm which is also the same as the $\mathcal{H}_\infty$ norm; in turn this is smaller than the ∞, ∞-induced norm, and all these numbers are upper-bounded by twice the sum of the Hankel singular values. ■



$$\sigma^2 = \begin{bmatrix} 9.9963e-001 \\ 9.9163e-001 \\ 9.2471e-001 \\ 6.8230e-001 \\ 3.1336e-001 \\ 7.7116e-002 \\ 1.0359e-002 \\ 8.4789e-004 \\ 4.5242e-005 \\ 1.5903e-006 \\ 3.6103e-008 \\ 5.0738e-010 \\ 4.1111e-012 \\ 1.6892e-014 \\ 7.5197e-017 \\ 3.2466e-017 \end{bmatrix}$$

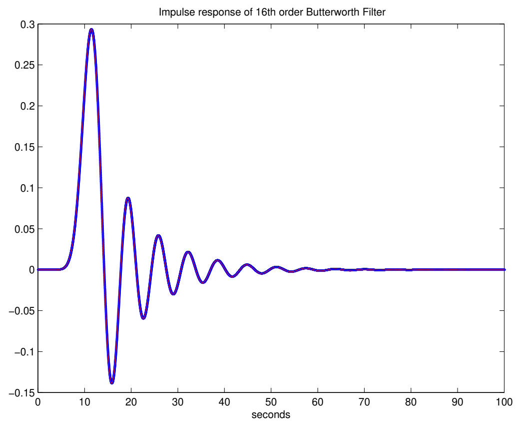


Figure 5: 16th order Butterworth filter: Nyquist diagram, Hankel singular values, impulse response.

4.4.4 Summary of norms

There are 3 quantities associated with a linear system $\mathbf{\Sigma}$ which can be used to define different norms. These are: the impulse response $h_{\mathbf{\Sigma}}$, the convolution operator $\mathcal{S}_{\mathbf{\Sigma}}$ and the Hankel operator $\mathcal{H}_{\mathbf{\Sigma}}$. One can now define (i) the $\mathcal{L}_1$ norm of $h_{\mathbf{\Sigma}}$ and (ii) the $\mathcal{L}_2$ norm of the same quantity $h_{\mathbf{\Sigma}}$. By means of the Hardy space $\mathcal{H}_2$, the latter is equal to the $\mathcal{H}_2$ norm of its transform $H_{\mathbf{\Sigma}}$ (which is the transfer function of $\mathbf{\Sigma}$).

One can also define *induced norms* of $\mathcal{S}_{\mathbf{\Sigma}}$ and $\mathcal{H}_{\mathbf{\Sigma}}$. Recall the definition (2.12). First we can define the $2,2$ norms, which according to (2.12) are obtained for $\alpha = \beta = 2$. According to lemma 2.1 this is equal to the largest singular value of $\mathcal{S}_{\mathbf{\Sigma}}$, $\mathcal{H}_{\mathbf{\Sigma}}$, respectively. Again because of the equivalence with the frequency domain the former is the $\mathcal{H}_\infty$ norm of the transfer function $H_{\mathbf{\Sigma}}$; the latter is the Hankel norm of $\mathbf{\Sigma}$.

Other induced norms of $\mathcal{S}$ are the $2,\infty$ ($\alpha = 2$, $\beta = \infty$) and the ∞,∞ or peak-to-peak norm ($\alpha = \beta = \infty$). It turns out that the former is closely related to the $\mathcal{L}_2$ norm of h, while the latter is the $\mathcal{L}_1$ norm of $h_{\mathbf{\Sigma}}$.

The interpretation of the induced norms of the convolution operator are as follows. The $2,\infty$ norm gives the largest magnitude of the output y (i.e. $\|y\|_\infty$) which is achievable with unit energy inputs u (i.e. $\|u\|_2 = 1$). The $2,2$ induced norm is the largest energy of the output y (i.e. $\|y\|_2$) given inputs u of unit energy (i.e. $\|u\|_2 = 1$). Finally, the ∞,∞ norm, also known as the *peak-to-peak norm*, is the largest magnitude of y (i.e. $\|y\|_\infty$) achieved with inputs u of unit largest magnitude (i.e. $\|u\|_\infty$=1). As for the Hankel norm, that is, the $2,2$ induced norm of the Hankel operator $\mathcal{H}$, it is the largest energy of *future* outputs y_+, caused by unit energy *past* inputs u_-. An interpretation of the singular values of this operator in the frequency domain is given by the fact that the Frobenius or Hilbert-Schmidt norm of $\mathcal{H}$ is equal to the area of the Nyquist diagram.

We conclude with some additional norms. Recall the definition of induced norms (2.12). Let, in addition, $\delta_{\max}(M)$ denote the largest element of the diagonal of the real matrix M. With this notation we have the following results.

$$\begin{array}{ll} ||\mathcal{S}||_{(2,1)-(2,2)-\text{ind}} = \sqrt{\lambda_{\max}(B^*\mathcal{P}B)} & ||\mathcal{S}||_{(1,1)-(2,2)-\text{ind}} = \sqrt{\delta_{\max}(B^*\mathcal{P}B)} \\ ||\mathcal{S}||_{(2,2)-(2,\infty)-\text{ind}} = \sqrt{\lambda_{\max}(C\mathcal{Q}C^*)} & ||\mathcal{S}||_{(2,2)-(\infty,\infty)-\text{ind}} = \sqrt{\delta_{\max}(C\mathcal{Q}C^*)} \\ ||\mathcal{S}||_{(1,1)-(\infty,\infty)-\text{ind}} = \sup_{t\geq 0} \|h(t)\|_\infty & ||\mathcal{S}||_{(2,1)-(2,\infty)-\text{ind}} = \sup_{t\geq 0} \sigma_{\max}(h(t)) \end{array}$$

Table 5 summarizes these facts and provides some comparisons between these norms.

4.5 The use of norms in control system design and model reduction

A general configuration for control system design is shown in figure 6. The (generalized) plant is denoted by P. It is influenced by two kinds of input or excitation functions: the *control input u* and the *exogenous input v*. The former (which may be vector valued, i.e. have more than one channels) is the input which is available for feedback. The latter represents the disturbance inputs. In addition, there are two kinds of output or obsevation functions, namely, the *to-be-controlled output* z and the *measured output* y. The problem now, consists of desigining a *feedback controller* K, acting between the measured output and the control input, so that some appropriate norm of the transfer from the exogenous inputs e to the to-be-controlled output z is minimized. Let the equations describing this configuration be:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} e \\ u \end{bmatrix}, \; u = Ky$$

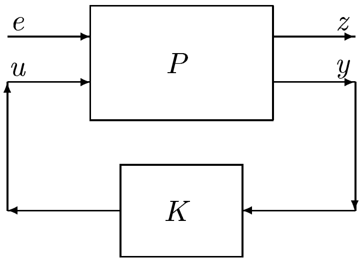


Figure 6: A general control design configuration

Eliminating u and y in the above equations we obtain

$$z = T_K e \;\text{ where }\; T_K = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

Thus, more precisely, the problem becomes the minimization of an appropriate norm of the above expression T_K over all admissible (i.e. internally stabilizing) controllers K. By means of the Youla-Kucera parametrization of all stabilizing controllers, T_K becomes an affine (and thus convex) function of a free parameter: $T_K(s) = R_1(s) - R_2(s)Q(s)R_3(s)$; $R_i(s)$, $i = 1, 2, 3$, are given, while $Q(s)$ is the unknown free parameter; all are stable, proper, rational. We will mention below three major categories of control system design. They are distinguished by the norm $\|T_K\|$ which is used.

- $\mathcal{H}_2$ **design**. This is traditionally known as Linear-Quadratic-Gaussian (LQG) design. Here we seek a controller K such that $\|T_K\|_{\mathcal{H}_2}$ is *minimized.* If e is a white noise input, the to-be-controlled output z is related as follows to the transfer function:

  $$\|T_K\|^2_{\mathcal{H}_2} = \lim_{T\to\infty} \left[\frac{1}{T}\int_0^\infty y^*(t)y(t)\,dt\right] = \|z\|^2_{\text{RMS}}.$$

  Thus the $\mathcal{H}_2$ designs seek to minimize the power of the output (also knwon as the RMS norm) of the output z.

  The solution involves two Riccati equations and the compensator is composed of an oberver (Kalman Filter) followed by state feedback.

- $\mathcal{H}_\infty$ **design**. In this case we seek to minimize the energy gain of the system, that is, the $\mathcal{H}_\infty$ norm of T_K. Therefore the energy of the output is minimized over all inputs of energy less than one.

  The solution in this case involves the solution of two Riccati equations as well.

- $\mathcal{L}_1$ **design**. In contrast to the previous two, this is a purely time-domain methodology. It seeks to minimize the $\mathcal{L}_1$ norm of T_K, that is, its peak-to-peak gain. Therefore, the amplitude of the to-be-controlled output z is minimized over all exogenous inputs of amplitude less than one.

  The solution is reduced to a *semi-infinite linear programming problem*. In contrast to the previous two designs, the compensator may have complexity by far larger that that of the to-be-controlled system.

4.5.1 Model reduction

Another instance where norms are essential, is *approximation* or *model reduction*.

Models of dynamical systems are useful primarily for two reasons: first for *simulation* and second for *control*. In many cases, any realistic model of a physical process will have high complexity, in other words it will require many state variables to be adequately described. The resulting complexity, i.e. number of first-order differential equations, is such that a simplification or model reduction will be needed in order to perform a simulation in an amount of time which is acceptable for the application at hand, or for the design of a low order controller which achieves desired objectives. Thus in all these cases *reduced-order models* are needed.

The systems considered are modeled by means of a set of first order coupled differential equations, together with a set of algebraic equations:

$$\mathbf{\Sigma}: \ \frac{dx(t)}{dt} = f(x(t), u(t)) y(t) = h(x(t), u(t)) \tag{4.10}$$

For simplicity we will use the following notation: $\mathbf{\Sigma} = (f, h)$, $u(t) \in \mathbb{R}^m$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$. In this setting, u is the input or excitation function, x is the state, and the function f describes the *dynamics* of $\mathbf{\Sigma}$; y on the other hand, is the output or set of observations and h describes the way that the observations are deduced from the state and the input. The *complexity* of $\mathbf{\Sigma}$ is defined as the number of states n.

The problem that will be addressed is to *simplify* or *approximate* $\mathbf{\Sigma}$ with another dynamical system $\hat{\mathbf{\Sigma}}$

$$\hat{\mathbf{\Sigma}} = (\hat{f}, \hat{h}), \ \ u(t) \in \mathbb{R}^m, \ \hat{x}(t) \in \mathbb{R}^k, \ \hat{y}(t) \in \mathbb{R}^p$$

The first requirement is that the number of states (i.e. the number of first order differential equations) of the approximant be less (or much less) than that of the original system

$$k \ll n$$

Obviously in the absence of additional ones, this requirement is easy to satisfy by mere elimination of equations and state variables. The difficulty arises by imposing additional constraints; one set of constraints usually attached to such an approximation problem is the following:

(1) Approximation error *small* - existence of *global error bound*
(2) Preservation of *stability*, *passivity*
(3) Procedure *computationally efficient*
(4) Procedure *automatic* based on error tolerance

The first constraint above is to be understood as follows. Let the input function u be fed into both $\mathbf{\Sigma}$ and $\hat{\mathbf{\Sigma}}$, and let the corresponding responses be y and $\hat{y}$, respectively. *Small approximation error* means that y is close to $\hat{y}$ in an appropriately defined sense and for a given set of inputs; one popular criterion in coming up with reduced-order approximants is the worse error $\|y - \hat{y}\|_2$, which should be minimized for all u (this gives rise to the $\mathcal{H}_\infty$ error criterion).

An important *special case*, is that of linear dynamical systems described by equations (3.13) and (3.14) and denoted for simplifuty by (3.15): $\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in \mathbb{R}^{(n+p)\times(n+m)}$. The problem in this case consists in approximating $\mathbf{\Sigma}$ with:

$$\hat{\mathbf{\Sigma}} = \left(\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right) \in \mathbb{R}^{(k+p)\times(k+m)}$$

where as before $k < n$ (or $k \ll n$) and the above four conditions are satisfied.

Approximation methods

Approximation methods can be cast into three broad categories: **(a)** SVD based methods, **(b)** Krylov based methods, **(c)** Iterative methods combining aspects of both the SVD and Krylov methods. In the sequel, we will give a brief overview of these approaches.

The **SVD-based approximation methods** have their roots in the Singular Value Decomposition and the resulting solution of the approximation of matrices by means of matrices of lower rank, which are optimal in the 2-norm (or more generally in unitarily invariant norms). The quantities which are important in deciding to what extent a given finite-dimensional operator can be approximated by one of lower rank are the so-called *singular values*; these (as already noted) are the square roots of the eigenvalues of the product of the operator and its adjoint. Important is that the ensuing error satisfies an apriori computable upper bound.

The question which arises is whether this result can be applied or extended to the case of dynamical systems. One straightforward way of applying it to a dynamical system described by (4.10) is as follows. Choose an input function and compute the resulting trajectory; collect samples of this trajectory at different times and compute the SVD of the resulting collection of samples. Then apply the SVD to the resulting matrix. This is a method which is widely used in computation involving PDEs; it is known as *Proper Orthogonal Decomposition (POD)*. The problem however in this case is that the resulting simplification heavily depends on the initial excitation function chosen, the time instances at which the measurements are taken; consequently, the singular values obtained are not system invariants. The advantage of this method is that it can be applied to high-complexity linear as well as nonlinear systems.

A different approach exists in the case of linear systems. First we observe that a linear time-invariant dynamical system $\mathbf{\Sigma}$, can be represented by means of an integral (convolution) operator; if in addition $\mathbf{\Sigma}$ is finite dimensional, then this operator has finite rank (and is hence compact). Consequently it has a set of finitely many non-zero singular values. Thus in principle, the SVD approximation method can be used to simplify such dynamical systems.

Indeed, as discussed earlier, there is a set of invariants, the *Hankel singular values*, which can be attached to every linear, constant, finite-dimensional system. These invariants play the same role for dynamical systems that the singular values play for constant finite-dimensional matrices. In other words, they determine the complexity of the reduced system and at the same time, they provide a global error bound for the resulting approximation. This gives rise to two model reduction methods, namely: **balanced model reduction** and **Hankel norm approximation**. It has been observed that the Hankel singular values of many systems decay extremely rapidly. Hence very low rank approximations are possible and accurate low order reduced models will result.

The first decisive results in this direction were obtained by Adamjan, Arov, and Krein. The theory as it is used today is due to Glover. Independently, Moore developed the concept of balancing; this led to the very popular method of approximation by balanced truncation.

The limitation of this approach comes from the fact that the computation of the Hankel singular values involves the solution of two linear matrix equations, the *Lyapunov equations*. Subsequently, the eigenvalues of the product of two positive definite matrices is required; these are the *reachability* and *observability* grammians.

The exact solution of such equations requires dense computations and therefore can roughly speaking be carried out for systems of modest dimension $n \approx \text{a few hundred}$.

A modification of this method has been developed for nonlinear systems. It is the method of *empirical grammians*. Its goal is to remedy the issues arising in POD methods, at the expense of added computational complexity.

A different set of approximation methods have been developed. They are *iterative* in nature and hence can be applied to very complex systems. These are the **Krylov-based** approximation methods, which do not rely on the computation of singular values. Instead they are based on *moment matching* of the *impulse response* of $\mathbf{\Sigma}$. Two widely used methods fall under this category, namely the **Lanczos** and the **Arnoldi** procedures, which were put forward by C. Lanczos in 1950 and by W.E. Arnoldi in 1951, respectively. These methods have been very influential in iterative eigenvalue computations and more recently in model reduction. Their drawbacks are that the resulting reduced order systems have no guaranteed error bound, stability is not necessarily preserved and some of them are not automatic.

Two leading efforts in this area are Padé via Lanczos (PVL) and multipoint rational interpolation. The PVL approach exploits the deep connection between the (nonsymmetric) Lanczos process and classical moment matching techniques, i.e. the fundamental connection of Krylov subspace projection methods in linear algebra to the partial realization problem in system theory.

The multipoint rational interpolation approach utilizes the rational Krylov method of Ruhe to provide moment matching of the transfer function at selected frequencies and hence to obtain enhanced approximation of the transfer function over a broad frequency range. These techniques have proven to be very effective. PVL has enjoyed considerable success in circuit simulation applications. Rational interpolation achieves remarkable approximation of the transfer function with very low order models. Nevertheless, there are shortcomings to both approaches. In particular, since the methods are local in nature, it is difficult to establish global error bounds. Heuristics have been developed that appear to work, but no global results exist. Secondly, the rational interpolation method requires selection of interpolation points. At present, this is not an automated process and relies on ad-hoc specification by the user.

This brings us to the third class of methods which seek to combine the best features of the SVD based methods and of the Krylov based methods. These are the **SVD-Krylov**-based approximation methods. The essential feature of the former is the solution of Lyapunov equations and the essential feature of the latter is the use of Krylov spaces (also known in system theory as *reachability-observability* subspaces). In particular it uses an iterative computation of projections onto these spaces. This gives rise to various approaches which have iterative Lyapunov solvers at their center.

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Basic Laplace transform properties		
Property	Time signal	$\mathcal{L}$-transform
Linearity	$af_1(t) + bf_2(t)$	$aF_1(s) + bF_2(s)$
Shifting in the s-domain	$e^{s_0 t} f(t)$	$F(s - s_0)$
Time scaling	$f(at), a > 0$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Convolution	$f_1(t) * f_2(t)$ $f_1(t) = f_2(t) = 0, t < 0$	$F_1(s)F_2(s)$
Differentiation in time	$\frac{d}{dt} f(t)$	$sF(s) - f(0^-)$
Differentiation in freq.	$-tf(t)$	$\frac{d}{ds} F(s)$
Integration in time	$\int_{0^-}^{t} f(\tau) d\tau$	$\frac{1}{s} F(s)$
Impulse	$\delta(t)$	1
Exponential	$e^{at}\mathbb{I}(t)$	$\frac{1}{s-a}$
Initial value theorem: $f(0^+) = \lim_{s\to\infty} sF(s)$		
Final value theorem: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$		

Table 1: Basic Laplace transform properties

The last 2 properties hold provided that $f(t)$ contains no impulses or higher-order singularities at $t = 0$.

Basic $\mathcal{Z}$-transform properties		
Property	Time signal	$\mathcal{Z}$-transform
Linearity	$af_1(t) + bf_2(t)$	$aF_1(z) + bF_2(z)$
Forward shift	$f(t-1)$	$z^{-1}F(z) + f(-1)$
Backward shift	$f(t+1)$	$zF(z) - zf(0)$
Scaling in freq.	$a^t f(t)$	$F(\frac{z}{a})$
Conjugation	$f^*(t)$	$F^*(z^*)$
Convolution	$f_1(t) * f_2(t)$ $f_1(t) = f_2(t) = 0, n < 0$	$F_1(z)F_2(z)$
Differentiation in freq.	$tf(t)$	$-z\frac{dF(z)}{dz}$
Impulse	$\delta(t)$	1
Exponential	$a^n \mathbb{I}(t)$	$\frac{z}{z-a}$
First difference	$f(t) - f(t-1)$	$(1 - z^{-1})F(z) - f(-1)$
Accumulation	$\sum_{k=0}^{n} f(t)$	$\frac{1}{1-z^{-1}}F(z)$
Initial value theorem: $f[0] = \lim_{z\to\infty} F(z)$		

Table 2: Basic $\mathcal{Z}$-transform properties

<table>
<tr><th colspan="2">I/O and I/S/O representation of continuous-time linear systems</th></tr>
<tr><th>I/O</th><th>I/S/O</th></tr>
<tr><td>variables: (u, y)
$Q\left(\frac{d}{dt}\right) y(t) = P\left(\frac{d}{dt}\right) u(t),$
$u(t), y(t) \in \mathbb{R}$</td><td>variables: (u, x, y)
$\frac{d}{dt}x(t) = Ax(t) + Bu(t),\ y(t) = Cx(t) + Du(t)$
$x(t) \in \mathbb{R}^n,\ \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in \mathbb{R}^{(n+p)\times(n+m)}$</td></tr>
<tr><th colspan="2">Impulse response</th></tr>
<tr><td>$Q\left(\frac{d}{dt}\right) h(t) = P\left(\frac{d}{dt}\right) \delta(t)$
$H(s) = \mathcal{L}(h(t)) = Q^{-1}(s)P(s)$</td><td>$h(t) = D\delta(t) + Ce^{At}B,\ t \geq 0$
$H(s) = D + C(sI - A)^{-1}B$</td></tr>
<tr><th colspan="2">Poles - characteristic roots</th></tr>
<tr><td>$\lambda_i,\ \det Q(\lambda_i) = 0,\ i = 1, \cdots, n$</td><td>$\det(\lambda_i I - A) = 0$</td></tr>
<tr><th colspan="2">Zeros</th></tr>
<tr><td>$z_i \in \mathbb{C}$: $\exists\, v_i \in \mathbb{C}^m$ satisfying
$H(z_i)v_i = 0$</td><td>$z_i \in \mathbb{C}$: $\exists\, w_i \in \mathbb{C}^n$, $v_i \in \mathbb{C}^m$, such that
$\begin{pmatrix} z_i I - A & -B \\ -C & -D \end{pmatrix} \begin{pmatrix} w_i \\ v_i \end{pmatrix} = 0$</td></tr>
<tr><td></td><th>Matrix exponential</th></tr>
<tr><td></td><td>$e^{At} = \sum_{i=0}^{\infty} \frac{t^k}{k!}A^k \Rightarrow \frac{d}{dt}e^{At} = Ae^{At}$
$\mathcal{L}(e^{At}) = (sI - A)^{-1}$</td></tr>
<tr><th colspan="2">Solution in the time domain</th></tr>
<tr><td>$y(t) = h(t) * u(t) \Rightarrow$
$y(t) = y_{zi}(t) + y_{zs}(t)$
where $y_{zi}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t}$
and $y_{zs}(t) = \int_{0^-}^{t} h(t-\tau)u(\tau)d\tau$</td><td>$x(t) = x_{zi}(t) + x_{zs}(t)$
$x(t) = e^{At}x(0^-) + \int_{0^-}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$
$y(t) = Ce^{At}x(0^-) + \int_{0^-}^{t} \underbrace{(D\delta(t-\tau) + Ce^{A(t-\tau)}B)}_{h(\cdot)} u(\tau)d\tau$
$\Rightarrow\ y(t) = Ce^{At}x(0^-) + \int_{0^-}^{t} h(t-\tau)u(\tau)dt$</td></tr>
<tr><th colspan="2">Solution in the frequency domain</th></tr>
<tr><td>$Y(s) = Q^{-1}(s)R(s) + H(s)U(s)$
$R(s)$: results from initial conditions</td><td>$X(s) = (sI - A)^{-1}x(0^-) + (sI - A)^{-1}BU(s)$
$Y(s) = C(sI - A)^{-1}x(0^-) + \underbrace{(D + C(sI - A)^{-1}B)}_{H(s)} U(s)$
$\Rightarrow\ Y(s) = C(sI - A)^{-1}x(0^-) + H(s)U(s)$</td></tr>
</table>

Table 3: I/O and I/S/O representation of continuous-time linear systems

I/O and I/S/O representation of discrete-time linear systems	
I/O	**I/S/O**
variables: (u, y) $Q(\sigma)\,y(t) = P(\sigma)\,u(t)$, $u(t), y(t) \in \mathbb{R}$	variables: (u, x, y) $\sigma x(t) = Ax(t) + Bu(t),\ y(t) = Cx(t) + Du(t)$ $x(t) \in \mathbb{R}^n$, $\left(\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right) \in \mathbb{R}^{(n+p)\times(n+m)}$
Impulse response	
$Q(\sigma)\,h(t) = P(\sigma)\,\delta(t)$ $H(z) = \mathcal{Z}(h(t)) = Q^{-1}(z)P(z)$	$h(0) = D,\ h(t) = CA^{t-1}B, t > 0$ $H(z) = D + C(zI - A)^{-1}B$
Poles - zeros: same as for $t \in \mathbb{R}$	
Exponents of a martix: $\mathcal{Z}(A^t) = (zI - A)^{-1}$	
Solution in the time domain	
$y(t) = h(t) * u(t) \Rightarrow$ $y(t) = y_{zi}(t) + y_{zs}(t)$ where $y_{zi}(t) = \sum_{i=1}^{n} c_i \lambda_i^t$ and $y_{zs}(t) = \sum_{\tau=0}^{t-1} h(t-\tau)u(\tau)$	$x(t) = x_{zi}(t) + x_{zs}(t)$ $x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{t-\tau-1}Bu(\tau)$ $y(t) = CA^t x(0) + \sum_{\tau=0}^{t-1} \underbrace{(D\delta(t-\tau) + CA^{t-\tau-1}B)}_{h(\cdot)} u(\tau)$ $y(t) = CA^t x(0) + \sum_{\tau=0}^{t-1} h(t-\tau)u(\tau)$
Solution in the frequency domain	
$Y(z) = Q^{-1}(z)R(z) + H(z)U(z)$ $R(z)$: results from initial conditions	$X(z) = (zI - A)^{-1}x(0) + (zI - A)^{-1}BU(z)$ $Y(z) = C(zI - A)^{-1}x(0) + \underbrace{(D + C(zI - A)^{-1}B)}_{H(z)} U(z)$ $\Rightarrow\ \ Y(z) = C(zI - A)^{-1}x(0) + H(z)U(z)$

Table 4: I/O and I/S/O representation of discrete-time linear systems

	Norms of linear systems		
	Time-Domain	**Frequency-Domain**	**Expression**
•	$\mathcal{L}_1$ norm of h: $\int_0^\infty \lVert h(t)\rVert_1 dt$		
•	$\mathcal{L}_2$ norm of h: $\left(\int_0^\infty \mathrm{trace}\,[h^*(t)h(t)]dt\right)^{1/2}$	$\mathcal{H}_2$ norm of H: $\left(\int_0^\infty \mathrm{trace}\,[H^*(-j\omega)H(j\omega)]d\omega\right)^{1/2}$	$\lVert\mathbf{\Sigma}\rVert_{\mathcal{H}_2}^2 = C\mathcal{P}C^* = B^*\mathcal{Q}B$
•	$\mathcal{L}_{2,2}$ induced norm of $\mathcal{S}$: $\sup_{u\neq 0}\frac{\lVert\mathcal{S}u\rVert}{\lVert u\rVert}$	$\mathcal{H}_\infty$ norm of H: $\sup_\omega \sigma_{\max} H(j\omega)$	$\lVert\mathbf{\Sigma}\rVert_{\mathcal{H}_\infty}$
•	$\mathcal{L}_{2,2}$ induced norm of $\mathcal{H}$ (Hankel norm): $\sup_{u_-\neq 0}\frac{\lVert\mathcal{H}u_-\rVert}{\lVert u_-\rVert}$ Hankel sing. values: $\sigma_i(\mathcal{H})$		$\lVert\mathbf{\Sigma}\rVert_{\mathrm{H}}^2 = \lambda_{\max}(\mathcal{P}\mathcal{Q})$ $\sigma_i^2(\mathbf{\Sigma}) = \lambda_i(\mathcal{P}\mathcal{Q})$
•	Hilbert-Schmidt norm of $\mathcal{H}$: $\sum_i \sigma_i^2(\mathcal{H})$ SISO case: $m = p = 1$	$\frac{1}{\pi}$ (area of Nyquist diagram)	

Relationships among norms

$$
\begin{array}{lllllll}
\lVert\mathcal{S}\rVert_{2,\infty} = \boxed{\sqrt{\lambda_{\max}(C\mathcal{P}C^*)}} & \leq & \lVert\mathcal{S}\rVert_{2,2} = \boxed{\sup_\omega \sigma_{\max}(H(j\omega))} & \leq & \lVert\mathcal{S}\rVert_{\infty,\infty} & \leq & 2\,\lVert\mathcal{H}\rVert_1 = \boxed{2\sum_i \sigma_i} \\
\wedge & & \vee & & \Vert & & \\
\lVert h\rVert_{\mathcal{L}_2} & & \lVert\mathcal{H}\rVert_{2,2} = \boxed{\sigma_1 = (\lambda_{\max}(\mathcal{P}\mathcal{Q}))^{1/2}} & & \lVert h\rVert_{\mathcal{L}_1} & & \\
\Vert & & & & & & \\
\lVert H\rVert_{\mathcal{H}_2} = \boxed{\sqrt{\mathrm{trace}\,(C\mathcal{P}C^*)}} & & & & & &
\end{array}
$$

Table 5: Norms of linear systems and their relationships