ELEC 302	HW #8	4/1/08

Due: 4/8/08 - 12 noon

Problem [1]. Given are two symmetric positive definite matrices (gramians) P, $Q \in \mathbb{R}^{n\times n}$. Consider the matrices PQ, $P^{\frac{1}{2}}QP^{\frac{1}{2}}$, $Q^{\frac{1}{2}}PQ^{\frac{1}{2}}$, and the pair of matrices (P, Q^{-1}). Show that the eigenvalues of the first three matrices and the generalized eigenvalues of the pair, are the same, and positive. Verify this in case the matrices are as in the previous homework assignement, namely:

$$P = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 2 \\ 0 & 2 & 6 \end{bmatrix}, \; Q = \begin{bmatrix} 7 & 6 & -2 \\ 6 & 6 & -2 \\ -2 & -2 & 1 \end{bmatrix} \;\Rightarrow\; Q^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & \frac{3}{2} & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Find the eigenvectors of each of the three matrices as well as the generalized eigenvectors of the given pair, and the relationships between them.

Problem [2]. Given is the following system:

$$\mathbf{\Sigma}: \; A = \begin{bmatrix} -0.746 & 0.006 & -0.999 & 0.0369 \\ -12.9 & -0.746 & 0.387 & 0 \\ 4.31 & 0.024 & -0.174 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0.0012 & 0.0092 \\ 6.05 & 0.952 \\ -0.416 & -1.76 \\ 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{\Sigma}$ describes the lateral motion of an aircraft, with state $x = [\beta \;\; \dot{\phi} \;\; r \;\; \phi]^*$, where β is the slip-angle, $\dot{\phi}$ is the roll rate, r is the yaw rate and ϕ is the roll angle. The control inputs are $u = [u_1 \;\; u_2]^*$, where u_1 is the aileron deflection and u_2 is the rudder deflection.

(a) Find a state transformation which will transform the system to a basis where e_1 is the state easiest to reach, e_2 the second-easiest, $\cdots$, and e_4 is the state which is least easy to reach. Hence reduce the system by eliminating the 2 states which are most difficult to reach What is the degree of observability of the states that have been retained?

(b) Find a state transformation which will transform the system to a basis where e_1 is the state easiest to observe, e_2 the second-easiest, $\cdots$, and e_4 is the state most difficult to observe. Hence reduce the systems by eliminating the two states which are most difficult to observe. What is the degree of reachability of the states that have not been retained?

(c) Find a balancing transformation, i.e. a transformation which will transform the system so that the state e_i, $i = 1, \cdots, 4$, is equally reachable and observable, and in addition e_i is more reachable/obervable than e_{i+1}, $i = 1, \cdots, 3$. Hence reduce the system by eliminating the two states which are most difficult to reach/observe.

(d) Plot the amplitude Bode diagrams of the original and the reduced systems.

Problem [3]. (1) Find a balancing transformation for the 6^{th} order continuous-time Butterworth filter, which we will denote by Σ. Hence compute the reduced systems of order $\hat{\Sigma}_k$, $k = 2, 4$, obtained by balanced truncation. (2) Find a transformation of the same system that will make all states equally reachable, i.e. $P = I_6$. Hence compute the reduced systems $\tilde{\Sigma}_k$, of order $k = 1, 2, 3, 4, 5$, obtained by truncation of the resulting realization. (3) Compare Σ, $\hat{\Sigma}_k$, $\tilde{\Sigma}_k$, $k = 2, 4$, by plotting the amplitude Bode diagrams. ■

Sample problems from past tests (will not be graded).

The following systems are given:

$$\Sigma_1: \; A_1 = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \; B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; C_1 = [0, \;\; 1]$$

$$\Sigma_2: \; A_2 = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \; B_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \; C_2 = [-\tfrac{1}{2}, \;\; \tfrac{1}{2}]$$

$$\Sigma_3: \; A_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \; B_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \; C_3 = [-\tfrac{1}{2}, \;\; \tfrac{1}{2}]$$

$$\Sigma_4: \; A_4 = \begin{pmatrix} -\frac{1}{4} & -1 \\ 1 & -\frac{1}{2} \end{pmatrix}, \; B_4 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; C_4 = [1, \;\; 1]$$

Problem [4]. (a) Is there a transformation T_1 which will transform system Σ_1 to system Σ_2? Is there a transformation T_2 which will transform system Σ_1 to system Σ_3? Is there a transformation T_3 which will transform system Σ_2 to system Σ_3? If the answer is yes to any of the above questions, find the corresponding transformation. If the answer is no, explain why not.

(b) Compute the Hankel singular values of Σ_1 and hence determine the balanced representation of this system. Subsequently compute the reduced system of dimension one determined by eliminating the state which is most difficult to reach and observe. ■

Problem [5]. Given are the matrices $K = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $L = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. Find the generalized eigenvalues of the pair (K, L). Subsequently, by means of the simultaneous diagonalization procedure find a transformation T such that

$$(TKT^*)(TLT^*) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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Problem [6]. Consider the system Σ_4.

(a) Compute the infinite reachability and the infinite observability grammians, and hence determine a balanced representation of this system.

(b) Furthermore compute a representation where all states are equally observable. In this representation, reduce the system to dimension one by eliminating the state which is most difficult to reach.

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