ELEC 302 HW #2: Solutions February 13, 2007

Problem [1]. The impulse response of an LTI system is $(e^{-t} - e^{-t}\cos t)\mathbb{I}(t)$, where $\mathbb{I}(t)$ denotes the unit step function. What is the step response? Find the input which gives rise to the output $(1 + e^{2t})\mathbb{I}(t)$, for the same LTI system. ■

Solution

$$h(t) = (e^{-t} - e^{-t}\cos(t))\mathbb{I}(t) \Rightarrow H(s) = \mathcal{L}(h) = \frac{1}{s+1} - \frac{s+1}{(s+1)^2+1}$$

$$u(t) = \mathbb{I}(t) \Rightarrow U(s) = \mathcal{L}(u) = \frac{1}{s}$$

So

$$Y(s) = H(s)\cdot U(s) = \frac{1}{s(s+1)} - \frac{s+1}{s[(s+1)^2+1]}.$$

$$Y(s) = \frac{s+1-s}{s(s+1)} - \frac{1}{2}\cdot\frac{2s+2}{s[(s+1)^2+1]}$$

$$Y(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{2}\cdot\frac{s^2+2s+2-s^2}{s[(s+1)^2+1]}$$

$$Y(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{2}\cdot\frac{(s+1)^2+1-s^2}{s[(s+1)^2+1]}$$

$$Y(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{2}\left[\frac{1}{s} - \frac{s}{(s+1)^2+1}\right]$$

$$Y(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{s+1-1}{(s+1)^2+1}$$

$$Y(s) = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\left[\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

Taking the inverse Laplace Transform of the above expression gives the step response

$$\boxed{y(t) = \mathcal{L}^{-1}(Y) = \left[\tfrac{1}{2} - e^{-t} + \tfrac{1}{2}\left(e^{-t}cos(t) - e^{-t}sin(t)\right)\right]\cdot\mathbb{I}(t)}$$

Given $y(t) = (1 + e^{2t})\mathbb{I}(t)$, we take the Laplace transform to get:

$$Y(s) = \frac{1}{s} + \frac{1}{s-2}.$$

Then

$$U(s) = \frac{Y(s)}{H(s)} = \frac{2\,(s-1)\,(s+1)\left[(s+1)^2+1\right]}{s\,(s-2)} = 2s^2 + 8s + 18 + 4\cdot\frac{8s-1}{s\,(s-2)}.$$

Partial fraction expansion for $\frac{8s-1}{s(s-2)}$ gives

$$\frac{8s-1}{s\,(s-2)} = \frac{A}{s} + \frac{B}{s-2} \text{ where } A = \frac{1}{2} \text{ and } B = \frac{15}{2} \text{ so}$$

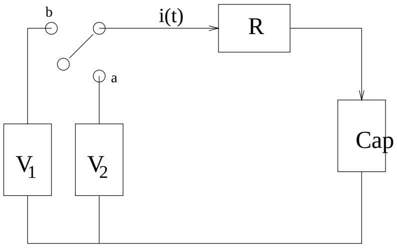
$$U(s) = 2s^2 + 8s + 18 + \frac{2}{s} + \frac{30}{s-2}$$

Taking the inverse Laplace Transform gives the required input

$$\boxed{u(t) = \mathcal{L}^{-1}(U) = 2\frac{d^2\delta(t)}{dt^2} + 8\frac{d\delta(t)}{dt} + 18\delta(t) + (2 + 30e^{2t}) \cdot \mathbb{I}(t)}$$

■

Problem [2]. Consider the circuit shown in the figure with $R = 1\Omega$ and $\mathrm{Cap} = 1F$. The switch is supposed to have been at position a for a long time (say, since $t = -\infty$). At $t = 0$, it goes to position b. Find the current $i(t)$, $t \geq 0$, for the following values of the two batteries:

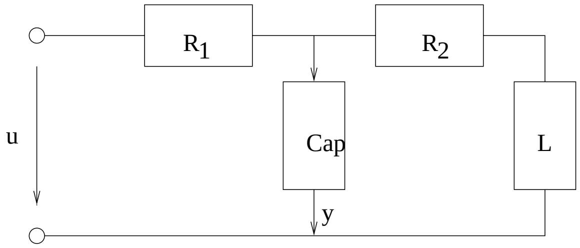


(i) $V_1 = 0V$, $V_2 = 1V$; (ii) $V_1 = 1V$, $V_2 = 0V$; (iii) $V_1 = 1V$, $V_2 = 1V$.
Using your answers for (i), (ii), (iii), argue that the current $i(t)$ can be considered as a sum of the circuit's zero-state response and zero-input response. ■

Solution
Case (i): $-e^{-t}$ because the capacitor is discharging
Case (ii): e^{-t} because the capacitor is charging
Case (iii): 0 because there is no change in the system

Thus case (iii) is the sum of case (i) (zero state) and case (ii) (zero input). ■

Problem [3]. Consider the RLC circuit shown in the figure. The notation is as follows: u is the input voltage, y the output current, x_1 is the voltage across the capacitor and x_2 the current through the inductor. **(a)** Write state and output equations and determine n, m, p and A, B, C, D. Write the i/o equation of this system in the form $q(\frac{d}{dt})y = p(\frac{d}{dt})u$ (determine $q(s)$ and $p(s)$). **(b)** Assume zero initial conditions and unit value for all elements. Compute the matrix exponential e^{At}, the state x, the transfer function and the impulse response. Hence



compute the step response. Identify the transient and steady state parts of each one of these functions. **(c)** From the transfer function of this system, or otherwise, deduce the differential equation relating y and u. ■

Solution

(a). From the circuit, we conclude that: m = number of inputs = 1, p = number of outputs = 1 and n = number of states = 2 (since there are two storage elements). We decide for the vector of states to be

$$x(t) = \begin{pmatrix} x_1(t) & = & v_{Cap}(t) \\ x_2(t) & = & i_L(t) \end{pmatrix}$$

The storage elements give the following relationships between the voltage drop and the current flow:

$$\text{inductor}: \; L\dot{i}_L = v_L \rightarrow L\dot{x}_2 = v_L$$

$$\text{capacitor}: \; i_{Cap} = Cap \;\; \dot{v}_{Cap} \rightarrow i_{Cap} = Cap \;\; \dot{x}_1$$

Kirchoff's Current Law gives

$$i_{R_1} = i_{Cap} + x_2 \rightarrow i_{R_1} = Cap \;\; \dot{x}_1 + x_2$$

Kirchoff's Voltage Law gives

$$u = R_1 \;\; i_{R_1} + x_1 = R_1\,(Cap \;\; \dot{x}_1 + x_2) + x_1 \rightarrow \dot{x}_1 = -\frac{1}{R_1 Cap}x_1 - \frac{1}{C}x_2 + \frac{1}{R_1 Cap}u \;\; \text{and}$$

$$x_1 = R_2 \;\; x_2 + v_L = R_2 \;\; x_2 + L\dot{x}_2 \rightarrow \dot{x}_2 = \frac{1}{L} \;\; x_1 - \frac{R_2}{L} \;\; x_2$$

The output equation is

$$y(t) = i_{Cap} = Cap \;\; \dot{x}_1 = -\frac{1}{R_1}x_1 - x_2 + \frac{1}{R_1}u$$

Therefore the matrices A, B, C, D are

$$A = \begin{bmatrix} -\frac{1}{R_1 Cap} & -\frac{1}{Cap} \\ \frac{1}{L} & -\frac{R_2}{L} \end{bmatrix}, \; B = \begin{bmatrix} \frac{1}{R_1 Cap} \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} -\frac{1}{R_1} & -1 \end{bmatrix}, \; D = \frac{1}{R_1}.$$

The transfer function of the system is

$$H(s)=\frac{1}{R_1}+\frac{\frac{1}{R_1Cap}\left[-\frac{1}{R_1}\left(s+\frac{R_1}{L}\right)-\frac{1}{L}\right]}{\left(s+\frac{1}{R_1Cap}\right)\left(s+\frac{R_2}{L}\right)+\frac{1}{Cap\ \ L}}$$

because

$$(sI-A)^{-1}=\frac{1}{\left(s+\frac{1}{R_1\ \ Cap}\right)\left(s+\frac{R_2}{L}\right)+\frac{1}{Cap}\frac{1}{L}}\begin{pmatrix} s+\frac{R_2}{L} & -\frac{1}{Cap} \\ \frac{1}{L} & s+\frac{1}{R_1Cap}\end{pmatrix}$$

and $H(s)=C\left(sI-A\right)^{-1}B+D$ so we obtain the expression above.

If $m=p=1$ then we can write the transfer function as a fraction $H(s)=\frac{p(s)}{q(s)}$. In our case we have

$$p(s)=\frac{1}{R_1}s^2+\frac{R_2}{R_1L}s$$

and

$$q(s)=s^2+s\left(\frac{R_2}{L}+\frac{1}{R_1Cap}\right)+\frac{R_2}{R_1\ \ Cap\ \ L}+\frac{1}{Cap\ \ L}$$

so the input/output equation of the system in differential form $q(\frac{d}{dt})y=p(\frac{d}{dt})u$ is

$$\ddot{y}(t)+\left(\frac{R_2}{L}+\frac{1}{R_1Cap}\right)\dot{y}(t)+\frac{R_2}{R_1\ \ Cap\ \ L}+\frac{1}{Cap\ \ L}y(t)=\frac{1}{R_1}\ddot{u}(t)+\frac{R_2}{R_1L}\dot{u}(t).$$

(b). Substituting for values we have the following state-space matrices

$$A=\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix},\ B=\begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C=\begin{bmatrix} -1 & -1 \end{bmatrix},\ D=1$$

To compute the matrix exponential, we can use either of the three approaches presented in class, for example the Laplace Transform Method: $e^{At}=\mathcal{L}^{-1}((sI-A)^{-1})$.

$$(sI-A)^{-1}=\begin{pmatrix} s+1 & 1 \\ -1 & s+1 \end{pmatrix}^{-1}=\frac{1}{(s+1)^2+1}\begin{pmatrix} s+1 & -1 \\ 1 & s+1 \end{pmatrix}$$

Therefore

$$e^{At}=\mathcal{L}^{-1}\begin{pmatrix} \frac{s+1}{(s+1)^2+1} & -\frac{1}{(s+1)^2+1} \\ \frac{1}{(s+1)^2+1} & \frac{s+1}{(s+1)^2+1} \end{pmatrix}=\begin{pmatrix} e^{-t}\cos(t) & -e^{-t}\sin(t) \\ e^{-t}\sin(t) & e^{-t}\cos(t) \end{pmatrix}$$

Using the formulas in the handout, we have

$$X(s)=(sI-A)^{-1}B=\begin{bmatrix} \frac{s+1}{(s+1)^2+1} \\ -\frac{1}{(s+1)^2+1} \end{bmatrix}\quad\Rightarrow\quad x(t)=\begin{bmatrix} e^{-t}\cos(t) \\ -e^{-t}\sin(t) \end{bmatrix}$$

The transfer function is

$$H(s) = 1 + \frac{-s-2}{(s+1)^2+1}$$

Thus the impulse response is

$$h(t) = \mathcal{L}^{-1}(H(s)) = \delta(t) - e^{-t}\sin t - e^{-t}\cos t, \;\; t \geq 0$$

while the step response in the s-domain is

$$R(s) = \frac{H(s)}{s} = \frac{1}{s} - \frac{s+1}{s\left((s+1)^2+1\right)} - \frac{1}{s\left((s+1)^2+1\right)}$$

In **Problem[1]** we found that

$$\frac{s+1}{s\left[(s+1)^2+1\right]} = \frac{1}{2}\left[\frac{1}{s} - \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}\right]$$

therefore

$$\frac{1}{s\left[(s+1)^2+1\right]} = \frac{s+1}{s\left[(s+1)^2+1\right]} - \frac{1}{(s+1)^2+1} = \frac{1}{2}\left[\frac{1}{s} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

so

$$R(s) = \frac{s+1}{(s+1)^2+1} \rightarrow r(t) = e^{-t}\cos t, \;\; t \geq 0$$

The steady-state parts of the state $x(t)$, impulse response $h(t)$ and step response $r(t)$ are

$$x_{ss}(t) = \lim_{t\to\infty}\begin{bmatrix} e^{-t}\text{cos}(t) \\ -e^{-t}\text{sin}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$h_{ss}(t) = \lim_{t\to\infty}\left(\delta(t) - e^{-t}\sin t - e^{-t}\cos t\right) = 0$$

$$r_{ss}(t) = \lim_{t\to\infty} e^{-t}\cos t = 0$$

so the transient responses are

$$x_{tr}(t) = x(t), h_{tr}(t) = h(t), r_{tr}(t) = r(t)$$

This is because all the above functions contain the exponential e^{-t} which decays to 0 as $t \to \infty$.

(c). The differential equation is

$$\ddot{y}(t) + \left(\frac{R_2}{L} + \frac{1}{R_1 Cap}\right)\dot{y}(t) + \frac{R_2}{R_1\;\,Cap\;\,L} + \frac{1}{Cap\;\,L}y(t) = \frac{1}{R_1}\ddot{u}(t) + \frac{R_2}{R_1 L}\dot{u}(t).$$

and when all elements have unit value, we get

$$\ddot{y}(t) + 2\dot{y}(t) + 2y(t) = \ddot{u}(t) + \dot{u}(t).$$

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Problem [4]. Put the following systems of differential equations in the form $\dot{x} = Ax + Bu$:

$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_2 + x_3,\\ \dot{x}_2 &= x_1 + x_2 - 2x_3,\\ \dot{x}_3 &= x_1 + x_2 - 2x_3.\end{aligned}$$

Compute e^{At} by using the definition of the matrix exponetial (i.e. computing the powers of A). What is $x(t)$ if $x_1(0) = x_2(0) = 0$ and $x_3(0) = 1$? Investigate the stability of this system. Finally, does there exist an initial condition $x(0)$ such that $x(t)$ remains bounded (i.e. does not go to infinity) as $t \to \infty$? If so, find all such initial conditions and explain their relationship with the eigenvectors of A. ■

Solution. The matrix A is given by

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{pmatrix}$$

Notice that

$$A^2 = \begin{pmatrix} 6 & 0 & -6 \\ -3 & 0 & 3 \\ -3 & 0 & 3 \end{pmatrix}$$

while

$$A^3 = \begin{pmatrix} -18 & 0 & 18 \\ 9 & 0 & -9 \\ 9 & 0 & -9 \end{pmatrix}$$

By induction, one can prove that

$$A^k = \begin{pmatrix} 2(-1)^k 3^{k-1} & 0 & 2(-1)^{k-1}3^{k-1} \\ (-1)^{k-1}3^{k-1} & 0 & (-1)^k 3^{k-1} \\ (-1)^{k-1}3^{k-1} & 0 & (-1)^k 3^{k-1} \end{pmatrix}, \quad k \geq 2$$

so the definition for the matrix exponential gives

$$e^{At} = I + \sum_{k\geq 1} A^k \frac{t^k}{k!} = \begin{pmatrix} \frac{2}{3}\sum_{k=0}^{\infty} \frac{(-3t)^k}{k!} + \frac{1}{3} & t & -\frac{2}{3}\sum_{k=0}^{\infty} \frac{(-3t)^k}{k!} + \frac{2}{3} - t \\ -\frac{1}{3}\sum_{k=0}^{\infty} \frac{(-3t)^k}{k!} + \frac{1}{3} & t+1 & \frac{1}{3}\sum_{k=0}^{\infty} \frac{(-3t)^k}{k!} - \frac{1}{3} - t \\ -\frac{1}{3}\sum_{k=0}^{\infty} \frac{(-3t)^k}{k!} + \frac{1}{3} & t & \frac{1}{3}\sum_{k=0}^{\infty} \frac{(-3t)^k}{k!} + \frac{2}{3} - t \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} \frac{2}{3}e^{-3t} + \frac{1}{3} & t & -\frac{2}{3}e^{-3t} + \frac{2}{3} - t \\ -\frac{1}{3}e^{-3t} + \frac{1}{3} & t+1 & \frac{1}{3}e^{-3t} - \frac{1}{3} - t \\ -\frac{1}{3}e^{-3t} + \frac{1}{3} & t & \frac{1}{3}e^{-3t} + \frac{2}{3} - t \end{pmatrix}$$

From the handout, we have that

$$x(t) = e^{At}x(0^-) = \left(\begin{array}{c} -\frac{2}{3}e^{-3t} + \frac{2}{3} - t \\ \frac{1}{3}e^{-3t} - \frac{1}{3} - t \\ \frac{1}{3}e^{-3t} + \frac{2}{3} - t \end{array} \right)$$

It follows that the system is unstable since for $t \to \infty$, $x(t)$ does not decay to 0.

The eigenvalues of A are $-3, 0$ and 0 so the matrix A does not have an Eigenvalue Decomposition, but a Jordan Block Decomposition with

$$A \underbrace{\left(\begin{array}{ccc} -2 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{array} \right)}_{V} = \underbrace{\left(\begin{array}{ccc} -2 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{array} \right)}_{V} \underbrace{\left(\begin{array}{ccc} -3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right)}_{D}$$

In order to obtain $x(t)$ bounded, we need to choose the initial condition $x(0^-)$ in the span of the first two columns of V. Therefore, choosing the initial condition as any linear combination of the two gives a bounded system

$$x(0^-) \in \left\{ \alpha \left(\begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right) + \beta \left(\begin{array}{c} -2 \\ 1 \\ 1 \end{array} \right), \alpha, \beta \in \Re \right\}$$

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