ELEC 302	HW #3: Solution	2/14/07

Problem [1]. Solution

(a)

1. KCL:

$$\begin{aligned} u &= C\dot{x}_2 \quad (1)\\ u &= x_1 + y \quad (2) \end{aligned}$$

2. KVL:

$$L\dot{x}_1 = Ry \quad (3)$$

So that the state and output equations become:

$$\begin{aligned} \dot{x}_1 &= -\frac{R}{L}x_1 + \frac{R}{L}u \quad (4)\\ \dot{x}_2 &= \frac{1}{C}u \quad (5)\\ y &= -x_1 + u, \quad (6) \end{aligned}$$

with system matrices:

$$\begin{aligned} A &= \begin{pmatrix} -\frac{R}{L} & 0 \\ 0 & 0 \end{pmatrix} \; B = \begin{pmatrix} \frac{R}{L} \\ \frac{1}{C} \end{pmatrix} \quad (7)\\ C &= \begin{pmatrix} -1 & 0 \end{pmatrix} \quad D = 1 \quad (8) \end{aligned}$$

(b) The system matrices are now:

$$\begin{aligned} A &= \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \; B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (9)\\ C &= \begin{pmatrix} -1 & 0 \end{pmatrix} \quad D = 1 \quad (10) \end{aligned}$$

We determine e^{At} using the inverse Laplace method:

$$\begin{aligned} \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] &= \mathcal{L}^{-1}[\begin{pmatrix} s+1 & 0 \\ 0 & s \end{pmatrix}^{-1}] = \mathcal{L}^{-1}[\frac{1}{s(s+1)}\begin{pmatrix} s & 0 \\ 0 & s+1 \end{pmatrix}] \quad (11)\\ &= \mathcal{L}^{-1}[\begin{pmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s} \end{pmatrix}] = \begin{pmatrix} e^{-t}\mathbb{I}(t) & 0 \\ 0 & \mathbb{I}(t) \end{pmatrix} \quad (12) \end{aligned}$$

With zero initial conditions and $u(t) = \mathbb{I}(t)$ we have:

$$\begin{aligned} \mathbf{x}(t) &= \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\, d\tau, \quad t \geq 0 \quad (13)\\ &= \int_0^t \begin{pmatrix} e^{-t+\tau}\mathbb{I}(-t+\tau) & 0 \\ 0 & \mathbb{I}(t-\tau) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} d\tau \quad (14)\\ &= \int_0^t \begin{pmatrix} e^{-t+\tau} \\ \mathbb{I}(t-\tau) \end{pmatrix} d\tau \quad (15)\\ &= \begin{pmatrix} 1 - e^{-t} \\ t \end{pmatrix} \mathbb{I}(t) \quad (16) \end{aligned}$$

From here we see that $x_{1ss} = x_1(t \to \infty) = 1$ while x_2 has no steady state since $x_2(t \to \infty) = \infty$. This was expected already from the fact that A has an eigenvalue 0, which means that the system is unstable. The transient response of x_1 is $x_1(t) - x_{1ss} = -e^{-t}$. Because $x_2(t) = t$ (unstable), x_2 does NOT have a transient part.

Similarly, assuming zero intial conditions,

$$\begin{aligned} y(t) &= \int_0^t (D\delta(t-\tau) + Ce^{A(t-\tau)}B)u(\tau)\,d\tau, \quad t \geq 0 && (17)\\ &= \int_0^t D\delta(t-\tau)\mathbb{I}(\tau)d\tau + \int_0^t (-1+e^{-t+\tau})\mathbb{I}(\tau)d\tau && (18)\\ &= D\mathbb{I}(t) + (-1+e^{-t})\mathbb{I}(t) = e^{-t}\mathbb{I}(t) && (19) \end{aligned}$$

With these results, you may want to check that indeed: $y = -x_1 + u$. $y_{ss} = y(t \to \infty) = 0 \Rightarrow$ the transient response of $y(t)$ is also $e^{-t}\mathbb{I}(t)$.

(c)

$$\begin{aligned} \mathbf{H}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} && (20)\\ &= 1 + \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 - \frac{1}{s+1} = \frac{s}{s+1} && (21) \end{aligned}$$

$\Rightarrow \ \delta(t) - e^{-t}\mathbb{I}(t)$. From (21) $H(s) = \frac{p(s)}{q(s)} \Rightarrow \ q(s)Y(s) = p(s)U(s) \Rightarrow \ (s+1)Y(s) = sU(s)$. So the differential equation is $\dot{y}(t) + y(t) = \dot{u}(t)$.

Problem [2]. Solution

(a) We have $q(s) = s^2 - 4$, $p(s) = 2s - 1$. Define the auxiliary variable w by means of $q(\frac{d}{dt})w(t) = u(t)$, then $y = p(\frac{d}{dt})w(t)$, and we can define the state variables as $x_1 = w$, $x_2 = \dot{w}$. Consequently

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 4 & 0 & 1 \\ \hline -1 & 2 & 0 \end{array}\right)$$

The state and output equations are:

$$\begin{aligned} \dot{x}_1 &= x_2 && (22)\\ \dot{x}_2 &= 4x_1 + u && (23)\\ y &= 2x_2 - x_1 && (24) \end{aligned}$$

(b)

1. **I/O** description:

   Assuming we don't have acces to the state (i.e. we have no knowledge of the system matrices), we find $h(t)$ from:

$$\begin{aligned} h(t) = \mathcal{L}^{-1}[H(s)] &= \mathcal{L}^{-1}\left[\frac{p(s)}{q(s)}\right] = \mathcal{L}^{-1}\left[\frac{2s-1}{s^2-4}\right] && (25)\\ &= \mathcal{L}^{-1}\left[\frac{3}{4}\frac{1}{s-2} + \frac{5}{4}\frac{1}{s+2}\right] && (26)\\ &= (\frac{3}{4}e^{2t} + \frac{5}{4}e^{-2t})\mathbb{I}(t) && (27) \end{aligned}$$

The system is not BIBO stable because the transfer function has one pole in the right half plane, i.e. $q(s)$ has a root at 2. Alternatively, one can check BIBO stabillity by determining whether $\int_0^\infty |h(t)|dt < \infty$. Clearly, our system is NOT BIBO stable because:

$$\int_0^\infty |h(t)|dt = \underbrace{\int_0^\infty \frac{3}{4}e^{2t}dt}_{\infty} + \int_0^\infty \frac{5}{4}e^{-2t}dt = \infty \tag{28}$$

The response to a step input is:

$$r(t) = h(t) \star \mathbb{I}(t) = \int_0^\infty \mathbb{I}(t-\tau)h(\tau)d\tau = \int_0^t h(\tau)d\tau = \frac{3}{8}e^{2t} - \frac{5}{8}e^{-2t} + \frac{1}{4} \tag{29}$$

2. **I/S/O** description

Method 1: time domain

Using the eigenvalue decomposition of A, we have:

$$e^{At} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} e^{2t}\mathbb{I}(t) & 0 \\ 0 & e^{-2t}\mathbb{I}(t) \end{pmatrix} \begin{pmatrix} \frac{\sqrt{5}}{2} & \frac{\sqrt{5}}{4} \\ \frac{-\sqrt{5}}{2} & \frac{\sqrt{5}}{4} \end{pmatrix} \tag{30}$$

$$= \begin{pmatrix} \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t} & \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t} \\ e^{2t} - e^{-2t} & \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t} \end{pmatrix} \mathbb{I}(t) \tag{31}$$

The impulse resonse is:

$$h(t) = Ce^{At}B = (\frac{3}{4}e^{2t} + \frac{5}{4}e^{-2t})\mathbb{I}(t) \tag{32}$$

Method 2: Laplace domain

The transfer function of the system is:

$$H(s) = C(sI - A)^{-1}B \tag{33}$$

$$= [-1 \quad 2] \begin{pmatrix} \frac{s}{(s-2)(s+2)} & \frac{1}{(s-2)(s+2)} \\ \frac{-4}{(s-2)(s+2)} & \frac{s}{(s-2)(s+2)} \end{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{2s-1}{(s-2)(s+2)} \tag{34}$$

$$\Rightarrow \quad h(t) = \mathcal{L}^{-1}[H(s)] = (\frac{1}{4}e^{-2t} - \frac{1}{4}e^{2t})\mathbb{I}(t) + 2(\frac{1}{2}e^{-2t} + \frac{1}{2}e^{-2t})\mathbb{I}(t) + 2(\frac{1}{4}e^{-2t} - \frac{1}{4}e^{2t})\delta(t) \tag{35}$$

$$= (\frac{3}{4}e^{2t} + \frac{5}{4}e^{-2t})\mathbb{I}(t) \tag{36}$$

as before.

In the I/S/O description, we check BIBO stability by looking at the eigenvalues of A. Here A, has one positive eigevalue, 2, so the system is not BIBO stable.

Assuming zero initial conditions, the response to a step input can now be found from:

$$R(s) = D + H(s)U(s) = H(s)\frac{1}{s} = \frac{-1}{s(s-2)(s+2)} + \frac{2}{(s-2)(s+2)} \tag{37}$$

$$= -\int_0^t (\frac{1}{4}e^{2\tau} - \frac{1}{4}e^{-2\tau})d\tau + 2(\frac{1}{4}e^{2\tau} - \frac{1}{4}e^{-2\tau}) \tag{38}$$

$$= \frac{3}{8}e^{2t} - \frac{5}{8}e^{-2t} + \frac{1}{4} \tag{39}$$

(c) We know:

$$y = Cx \tag{40}$$

$$\dot{y} = C\dot{x} = CAx + CBu \tag{41}$$

So we can solve for x from:

$$\begin{bmatrix} C \\ CA \end{bmatrix} x = \begin{bmatrix} y \\ \dot{y} - CBu \end{bmatrix} \Rightarrow \tag{42}$$

$$\begin{bmatrix} -1 & 2 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} - 2u \end{bmatrix} \Rightarrow \tag{43}$$

$$x_1 = \frac{2}{15}\dot{y} + \frac{1}{15}y - \frac{4}{15}u \tag{44}$$

$$x_2 = \frac{1}{15}\dot{y} + \frac{8}{15}y - \frac{2}{15}u \tag{45}$$

(d) Using the results from **(c)** and $\dot{y}(0) = 0, \quad y(0) = 1$ we have $x_1(0) = \frac{1}{15}, \quad x_20 = \frac{8}{15}$, so that:

$$x_{zi}(t) = e^{At}x(0^-) = \begin{bmatrix} \frac{1}{6}e^{2t} - \frac{1}{10}e^{-2t} \\ \frac{1}{3}e^{2t} - \frac{1}{5}e^{-2t} \end{bmatrix} \tag{46}$$

$$x_{xs}(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{47}$$

$$= \int_0^t \begin{bmatrix} (\frac{1}{4}e^{2t-2\tau} - \frac{1}{4}e^{-2t+2\tau})e^{-2\tau} \\ (\frac{1}{2}e^{2t-2\tau} + \frac{1}{2}e^{-2t+2\tau})e^{-2\tau} \end{bmatrix} d\tau \tag{48}$$

$$= \begin{bmatrix} -\frac{1}{4}te^{-2t} + \frac{1}{16}e^{2t} - \frac{1}{16}e^{-2t} \\ \frac{1}{2}te^{-2t} + \frac{1}{8}e^{2t} - \frac{1}{8}e^{-2t} \end{bmatrix} \tag{49}$$

$$x(t) = x_{zi}(t) + x_{zs}(t) \tag{50}$$

Due to the positive exponential terms in $x(t)$, $x_{ss} = lim_{t \leftarrow \infty} x(t) = \infty$ so there is no steady state part in $x(t)$. The transient part of $x(t)$ consists of all terms of the form e^{-2t} and te^{-2t}.

Similarly,

$$y_{zi}(t) = Ce^{At}x(0^-) = [-1 \;\; 2] \begin{bmatrix} \frac{1}{6}e^{2t} - \frac{1}{10}e^{-2t} \\ \frac{1}{3}e^{2t} - \frac{1}{5}e^{-2t} \end{bmatrix} = \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t} \tag{51}$$

$$y_{zs}(t) = \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau \tag{52}$$

$$= [-1 \;\; 2] \begin{bmatrix} -\frac{1}{4}te^{-2t} + \frac{1}{16}e^{2t} - \frac{1}{16}e^{-2t} \\ \frac{1}{2}te^{-2t} + \frac{1}{8}e^{2t} - \frac{1}{8}e^{-2t} \end{bmatrix} = \frac{3}{16}e^{-2t} + \frac{3}{16}e^{2t} + \frac{5}{4}te^{-2t} \tag{53}$$

$$y(t) = y_{zi}(t) + y_{zs}(t) \tag{54}$$

For the same arguments as above, $y(t)$ has no steady state part and the transient part of $y(t)$ consists of all terms of the form e^{-2t} and te^{-2t}.

Problem [3]. Given is $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$. Is the associated system $\dot{x}(t) = Ax(t)$ asymptotically stable? If not, find all initial conditions $x(0)$ such that $x(t)$, $t > 0$, remains bounded. What is the relationship between these initial conditions and the eigenvectors of A? ■

Solution. The eigenvalues of A are -2 and 1 with algebraic multiplicity 2. The associated system is not asymptotically stable because A has eigenvalues in the right half plane. The solution is: $x(t) = e^{At}x(0^-)$ so a proper choice of initial conditions will make $x(t)$ bounded.

The matrix exponantial in this case is:

$$e^{At} = \mathcal{L}^{-1}[(sI-A)^{-1}] = \mathcal{L}^{-1}\left[\frac{1}{(s+2)(s-1)^2}\begin{pmatrix} (s-1)^2 & 0 & 0 \\ 0 & (s+2)(s-1) & 0 \\ 0 & 2(s+2) & (s+2)(s-1) \end{pmatrix}\right] \quad (55)$$

$$= \begin{pmatrix} \frac{1}{s+2} & 0 & 0 \\ 0 & \frac{1}{s-1} & 0 \\ 0 & \frac{2}{(s-1)^2} & \frac{1}{s-1} \end{pmatrix} \quad (56)$$

$$= \begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 2te^{t} & e^{t} \end{pmatrix} \quad (57)$$

The solution is:

$$x(t) = \begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 2te^{t} & e^{t} \end{pmatrix} x(0^-) \Rightarrow$$

For a stable $x(t)$, we must eliminate the positive exponential components by setting $x(0^-) = [\ a\ 0\ 0\ ]^T$, where a is arbitrary. Since the eigenvector of A corresponding to the stable eigenvalue -2 is e_1, we see that the initial condition has to be a multiple of this eigenvector. In general, the initial condition that stabilizes the solution has to be chosen in the span of the eigenvectors of A corresponding to the stable eigenvalues.

Problem [4]. Solution

$det(\lambda I - A) = \lambda^3 \Rightarrow$ A has one 0 eigenvalue with algebraic multiplicity 3, so it is not diagonalizable (it can only be decomposed to Jordan Normal form).

We compute $e^{At} = I + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{3!} + \cdots$

$$A = \begin{bmatrix} -4 & -2 & -2 \\ 3 & 2 & 1 \\ 4 & 2 & 2 \end{bmatrix} \quad (58)$$

$$A^2 = \begin{bmatrix} 2 & 0 & 2 \\ -2 & 0 & -2 \\ -2 & 0 & -2 \end{bmatrix} \quad (59)$$

$$A^3 = \mathbf{0} \Rightarrow \quad (60)$$

$$e^{At} = I + At + A^2\frac{t^2}{2} = \begin{pmatrix} 1-4t+t^2 & -2t & -2t+t^2 \\ 3t-t^2 & 1+2t & t-t^2 \\ 4t-t^2 & 2t & 1+2t-t^2 \end{pmatrix} \Rightarrow \quad (61)$$

$$h(t) = Ce^{At}B = [a\ b\ c]\begin{bmatrix} 1-4t \\ 1+4t \\ -1+4t \end{bmatrix} \quad (62)$$

$$= (a+b-c) + 4(-a+b+c)t \quad (63)$$

For $h(t)$ to be a step function, we need to eliminate the t term, by setting $a = b+c$. This will make $h(t) = 2b\mathbb{I}(t)$, i.e. a step function of amplitude $2b$. The dimension of the solution set is 1, i.e. the dimension of the left nullspace of A. We have infinitely many solutions, due to the arbitrary value of b.

Alternative method: Jordan Normal Form

Matrix A has an eigevalue 0 with multiplicity 3, so it does not have an EVD. Alternatively, A has a Jordan block, i.e. it can be decomposed into:

$$A = VJV^{-1} \quad (64)$$

$$J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \quad (65)$$

where J is the *Jordan block* formed by the repeating eigenvalues of A on the diagonal (in our case, $\lambda = 0$) and 1's on the superdiagonal.

The matrix $V = [v1,\ v2,\ v3]$ is found as follows. v1 is the eigenvector of A corresponding to the eigenvalue 0, i.e. v1 is in the nullspace of $(A - 0 \cdot I)$. We compute the generalized eigenvectors $v2$ and $v3$ by:

$$(A - 0 \cdot I)v1 = 0 \Rightarrow v1 = [\ -1,\ \ 1,\ \ 1\ ]^T \tag{66}$$
$$(A - 0 \cdot I)v2 = v1 \Rightarrow v2 = [\ 0,\ \ 1/2,\ \ 0\ ]^T \tag{67}$$
$$(A - 0 \cdot I)v3 = v2 \Rightarrow v3 = [\ -1/2,\ \ 1,\ \ 0\ ]^T \Rightarrow \tag{68}$$

Also notice that since:

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow J^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ J^3 = \mathbf{0} \Rightarrow \tag{69}$$
$$e^{Jt} = I + Jt + J^2\frac{t^2}{2} + \mathbf{0} \tag{70}$$
$$= \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \tag{71}$$

Using this we find:

$$e^{At} = Ve^{Jt}V^{-1} \tag{72}$$
$$= \begin{pmatrix} -1 & 0 & -1/2 \\ 1 & 1/2 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 4 & 2 & 2 \\ -2 & 0 & -2 \end{pmatrix} \tag{73}$$
$$= \begin{pmatrix} 1 - 4t + t^2 & -2t & -2t + t^2 \\ 3t - t^2 & 1 + 2t & t - t^2 \\ 4t - t^2 & 2t & 1 + 2t - t^2 \end{pmatrix} \tag{74}$$

as before.

Note: If A had one eigenvalue λ_1 with algebraic multiplicity 1 and an eigenvalue λ_2 with algebraic multiplicity 2, J would have the structure:

$$J = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda_2 & 1 \end{pmatrix} \tag{75}$$

To find V, we proceed similarly: v1 is now the eigenvector for eigenvalue λ_1, v2 is the eigenvector for eigenvalue λ_2 and v3 is found from $(A - \lambda_2 I)v_3 = v_2$.