ELEC 302 Solutions: HW #4: Solution 2/14/07

Problem [1]. Consider the system given by:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

(a) Find the transfer function of this system and hence the differential equations describing the system in terms of inputs and outputs (i.e. no states)

(b) Determine controllability from u_1,u_2 and u. If any of these systems is not controllable, find a basis for the controllable space.

Solution

(a)

$$\begin{aligned} H(s) = C(sI-A)^{-1}B &= \begin{bmatrix} \frac{1}{s^3+s^2+2s+1} & \frac{s^3+s+1}{s^4+2s^3+3s^2+3s+1} \\ \frac{s^2+s+2}{s^3+s^2+2s+1} & \frac{1}{s^3+s^2+2s+1} \end{bmatrix} && (2) \\ &= \begin{bmatrix} \frac{p_{11}(s)}{q_{11}(s)} & \frac{p_{12}(s)}{q_{12}(s)} \\ \frac{p_{21}(s)}{q_{11}(s)} & \frac{p_{22}(s)}{q_{11}(s)} \end{bmatrix} && (3) \end{aligned}$$

where $q_{12(s)} = q_{11}(s)(s+1)$.
Since the system has two inputs and two outputs, we have:

$$\begin{aligned} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} &= H(s) \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} && (4) \\ &= \begin{bmatrix} \frac{p_{11}(s)}{q_{11}(s)} & \frac{p_{12}(s)}{q_{12}(s)} \\ \frac{p_{21}(s)}{q_{11}(s)} & \frac{p_{22}(s)}{q_{11}(s)} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} \Rightarrow && (5) \end{aligned}$$

We now treat each output independently to find the differential equations relating them to both inputs, u_1 and u_2. For y_1 we have:

$$\begin{aligned} (s+1)q_{11}(s)Y_1(s) &= (s+1)p_{11}(s)U_1(s) + p_{12}(s)U_2(s) \Rightarrow && (6) \\ (s^4+2s^3+3s^2+3s+1)Y_1(s) &= (s+1)U_1(s) + (s^3+s+1)U_2(s) \Rightarrow && (7) \\ y_1^{(4)} + 2y_1^{(3)} + 3y_1^{(2)} + 3\dot{y}_1 + y_1 &= \dot{u}_1 + u_1 + u_2^{(3)} + \dot{u}_2 + u_2 && (8) \end{aligned}$$

Similarly, for y_2:

$$\begin{aligned} q_{11}(s)Y_2(s) &= p_{21}(s)U_1(s) + p_{22}(s)U_2(s) \Rightarrow && (9) \\ (s^3+s^2+2s+1)Y_2(s) &= (s^2+s+2)U_1(s) + U_2(s) \Rightarrow && (10) \\ y_2^{(3)} + y_2^{(2)} + 2\dot{y}_2 + y_2 &= u_1^{(2)} + \dot{u}_1 + 2u_1 + u_2 && (11) \end{aligned}$$

(b) *Controllability from* u:

$$\mathcal{R}_4(A,B) = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 & 0 & -2 \\ 0 & 1 & -1 & 0 & 0 & -2 & 2 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 1 & -1 & 1 \end{bmatrix} \tag{12}$$

which, easily seen, has 4 linearly independent columns $\Rightarrow$ $rank(\mathcal{R}_4(A,B)) = 4$ $\Rightarrow$ $\mathbb{X}^{\text{contr}} = \mathbf{im}\mathcal{R}_4(A,B) = \mathbf{R}^4$ so the system is completely controllable from u.

Controllability from u_1: We construct the new controllability matrix using the first column of B only:

$$\mathcal{R}_4(A,B(:,1)) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{13}$$

Note that we do not actually need to recompute the controllability matrix $\mathcal{R}_4(A,B(:,1))$, since it can be extracted directly from $\mathcal{R}_4(A,B)$ by eliminating even columns. Since:

$$\begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix} = -\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

$\mathcal{R}_4(A,B(:,1))$ has only 3 linearly independent columns so its rank is 3 and the system is not completely controllable from u_1. The dimension of the controllable subspace is 3 and a basis for $\mathbb{X}^{\text{contr}}$ is:

$$\left(\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right) \tag{15}$$

To show that the three vectors in (15) are linearly idependent, take the matrix:

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{16}$$

and check whether one of the 3×3 sumbatrices resulting from a row elimination from (16) has a non-zero determinant. Indeed, if we eliminate row 3, the determinant of the remaining 3×3 matrix is -1, so the three vectors in (15) are linearly independent and form a basis for the controllable subspace.

Controllability from u_2: We now take the second column if be only and form:

$$\mathcal{R}_4(A,B(:,2)) = \begin{bmatrix} 0 & 1 & 0 & -2 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \tag{17}$$

All 4 columns are linearly independent so $rank(\mathcal{R}_4(A,B(:,2))) = 4$ and the system is completely controllable from u_2.

Problem [2]. Consider the inverted penduli system given in the *Case Studies* for the following values of the parameters: $M = 2$, $m = 1$, $\ell_1 = 1$, $\ell_2 = 1$, $g = 10$.

(a) Is the system stable? (b) Is the system controllable? Find the dimension and a basis for the controllable space. Is it possible to transfer each one of the initial states

$$\tilde{x} = [1\ \ 0\ \ 0\ \ 0]^T, \ \ \hat{x} = [1\ \ 1\ \ 0\ \ 0]^T, \ \ \bar{x} = [0\ \ 0\ \ 1\ \ 1]^T$$

to the zero state? Furthermore, is it possible to transfer the state of the system from the initial state $\hat{x}$ to the final state $\bar{x}$? Justify your answers. ■

Solution.

(a) From given values of the system parameters we have: $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -15 & -5 & 0 & 0 \\ -5 & -15 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

The eigenvalues of matrix A are $\pm 4.4721i$ and $\pm 3.1623i$, so since there are eigenvalues with real part 0, the system is *not stable*.

(b) Throughout the rest of the problem, we use the equivalence between the reachability and controllability concepts. The reachability (controllability) matrix is:

$$R_4(A,B) = \begin{bmatrix} B, & AB, & A^2B, & A^3B \end{bmatrix} = \begin{bmatrix} 0 & -0.5 & 0 & 10 \\ 0 & -0.5 & 0 & 10 \\ -0.5 & 0 & 10 & 0 \\ -0.5 & 0 & 10 & 0 \end{bmatrix}$$

So we see that rank $R_4(A,B) = 2$, since $R_4(A,B)$ has only 2 linearly independent columns; it follows that $\operatorname{im} R_4(A,B) = \mathbf{im}([v_1,\ v_2])$, where $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

$\bar{x} = v_1$, $\hat{x} = v_2$ while $\tilde{x}$ is not in the span of the basis vectors of the reachable subspace. So $\bar{x}$ and $\hat{x}$ are reachable states while $\tilde{x}$ is not.

In general, to determine whether it is possible to transfer the state of the system from an initial state $\hat{x}$ at time 0 to a final state $\bar{x}$ at time T, we need to determine whether there exists an input u such that:

$$\bar{x} = e^{AT}\hat{x} + \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau \ \Rightarrow \tag{18}$$

according the the reachability definition, we have to determine whether $\bar{x} - e^{AT}\hat{x} \in \mathbb{X}^{\text{reach}}$.

We know $\bar{x} \in \mathbb{X}^{\text{reach}}$ and $e^{AT}\hat{x} \in A\mathbb{X}^{\text{reach}} \subset \mathbb{X}^{\text{reach}}$ (recall that $e^{AT} = I + AT + A^2\frac{T^2}{2} + \cdots$). Using this and the fact that $\mathbb{X}^{\text{reach}}$ is a linear subspace, we conclude that $\bar{x} - e^{AT}\hat{x} \in \mathbb{X}^{\text{reach}}$, so there is a trajecotry from $\hat{x}$ to $\bar{x}$. Furthermore, since $\tilde{x}$ is not reachable, it cannot be tranferred to the zero state.

Problem [3]. Consider the RLC system given in the *Case Studies* with the following values of the parameters: $C_1 = C_2 = 1$, $R = R_1 = R_2 = 1$, $L_1 = L_2 = 1$. Consider also the states:

$$\hat{x} = [0\ \ 1\ \ 0\ \ 0]^T,\ \tilde{x} = [0\ \ 0\ \ 0\ \ 1]^T$$

(a) Is the system stable? Is the system controllable? Find the dimension and a basis for the controllable space.

(b) Explain whether there exists an input $\hat{u}$ which will transfer the state of the system from 0 to $\hat{x}$. Is it possible to transfer the state of the system from $\hat{x}$ to $\tilde{x}$? Justify your answer. ■

Solution.

(a)

From the given circuit we have:

$\dot{x_1} = -x_1 - x_2 - x_3 + u_1$

$\dot{x_2} = x_1 - x_2$

$\dot{x_3} = -x_1 - x_3 - x_4 + u_2$

$\dot{x_4} = x_3 - x_4$

Thus we have $A = \begin{bmatrix} -1 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. The eigenvalues of A are $\lambda_{1,2} = -\frac{3}{2} \pm \frac{\sqrt{3}}{2} i$ and $\lambda_{3,4} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$, and hence the system is stable. The reachability matrix is:

$$R_4(A,B) = \begin{bmatrix} B, & AB, & A^2B, & A^3B \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & -1 & 1 & 2 & -1 & -2 \\ 0 & 0 & 1 & 0 & -2 & -1 & 3 & 3 \\ 0 & 1 & -1 & -1 & 2 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 & -1 & -2 & 3 & 3 \end{bmatrix}$$

Clearly the first 4 columns of this matrix are linearly independent and thus the system is completely reachable given both inputs u_1 and u_2. A basis for the reachable space is any basis for $\mathbb{R}^4$.

(b) Since the system is reachable there exist inputs that will drive the system from the zero state to any state in $\mathbb{R}^4$.

If we wish to drive the system from state x_1 at $t = 0$ to state x_2 at $t = T$, there must exist and input $u(t)$ such that

$$x_2 = e^{AT} x_1 + \int_0^T e^{A(T-\tau)} B u(\tau)\, d\tau \;\Rightarrow\; x_2 - e^{AT} x_1 \in \mathbb{X}^{\text{reach}}$$

Therefore since $A\mathbb{X}^{\text{reach}} \subset \mathbb{X}^{\text{reach}}$ the above relationship implies that any two states in the reachable subspace can be joined by a state trajectory for an appropriate input. In this case the reachable subspace is the whole space and thus the given states can indeed be joined by a state trajectory.