ELEC 302 HW #9: Solutions

For questions, comments or typos: `cosmin.ionita[at]rice.edu`

Problem [1]. Clearly

$$PQ = P^{\frac{1}{2}}\left(P^{\frac{1}{2}}QP^{\frac{1}{2}}\right)P^{-\frac{1}{2}},$$
$$PQ = Q^{-\frac{1}{2}}\left(Q^{\frac{1}{2}}PQ^{\frac{1}{2}}\right)Q^{\frac{1}{2}}.$$

As eigenvalues are preserved under similarity transformations, it follows that the matrices PQ, $P^{\frac{1}{2}}QP^{\frac{1}{2}}$, $Q^{\frac{1}{2}}PQ^{\frac{1}{2}}$ have the same eigenvalues.

Let (λ, v) be a generalized eigen pair of (P, Q^{-1}). Then

$$(P - \lambda Q^{-1})v = 0.$$

Furthermore, there exists a vector w such that v can be written as $v = Qw$. Thus,

$$(PQ - \lambda I)w = 0,$$

and λ is also an eigenvalue of PQ, with w its corresponding eigenvector.

We now want to show that all the eigenvalues are also positive. We will show that $P^{\frac{1}{2}}QP^{\frac{1}{2}}$ is symmetric positive-definite.

As P is symmetric positive-definite, its square-root $P^{\frac{1}{2}}$ is symmetric positive-definite. Thus $P^{\frac{1}{2}}QP^{\frac{1}{2}}$ is a product of symmetric matrices and thus it is a symmetric matrix itself. To show it is also positive-definite, let $v \in \mathbb{R}^n$, and $w = P^{\frac{1}{2}}v$. Then

$$v^T P^{\frac{1}{2}}QP^{\frac{1}{2}}v = v^T(P^{\frac{1}{2}})^T QP^{\frac{1}{2}}v = w^T Qw > 0,$$

because Q is a positive-definite matrix. Therefore, $P^{\frac{1}{2}}QP^{\frac{1}{2}}$ is positive-definite, which implies that all of its eigenvalues are greater than 0.

Eigenvectors of PQ

```
        0    0.7071         0
   1.0000   -0.7071         0
        0         0    1.0000
```

Eigenvectors of $P^{\frac{1}{2}}QP^{\frac{1}{2}}$

```
  -0.8508    0.5256    0.1509
   0.5174    0.8376    0.4083
  -0.0920   -0.1490    0.9003
```

Eigenvectors of $Q^{\frac{1}{2}}PQ^{\frac{1}{2}}$

```
  -0.8225    0.5688   -0.4347
   0.5621    0.8128   -0.5224
  -0.0870   -0.1258    0.7336
```

Eigenvectors of (P, Q^{-1})

```
   -1.0000   -1.0000    0.9451
         0   -1.0000    0.9451
         0    0.3023    1.0000
```

If v is an eigenvector of PQ, then $P^{-\frac{1}{2}}v$ is an eigenvector of $P^{\frac{1}{2}}QP^{\frac{1}{2}}$, $Q^{\frac{1}{2}}v$ is an eigenvector of $Q^{\frac{1}{2}}PQ^{\frac{1}{2}}$ and Qv is an eigenvector of (P, Q^{-1}).