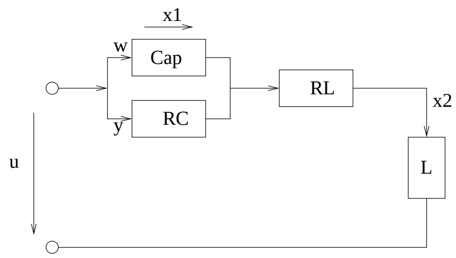
Problem [1].



Consider the circuit shown in the figure. Two systems are defined: $\Sigma_y \; : \; u; x_1, x_2; y$ and $\Sigma_w : \; u; x_1, x_2; w$. The input u, the state variables x_1, x_2 and the two different outputs (measurements) y, w are as shown.

(a) Write state equations using the state variables x_1, x_2, and determine the corresponding A, B. For each one of the systems Σ_y, Σ_w, determine the output equations, that is determine C, D.

Solution. KCL:

$$w + y = x_2 \tag{1}$$
$$w = C\dot{x}_1 \tag{2}$$

KVL:

$$u = x_1 + R_L x_2 + L\dot{x}_2 \tag{3}$$
$$yR_L = x_1 \tag{4}$$

From this we write the state and output equations:

$$\dot{x}_1 = -\frac{1}{CR_C}x_1 + \frac{1}{C}x_2 \tag{5}$$
$$\dot{x}_2 = -\frac{1}{L}x_1 - \frac{R_L}{L}x_2 + \frac{1}{L}u \tag{6}$$
$$y = \frac{1}{R_C}x_1 \tag{7}$$
$$w = -\frac{1}{R_C}x_1 + x_2 \tag{8}$$

The system matrices result:

$$A = \begin{bmatrix} -\frac{1}{CR_C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R_L}{L} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} \tag{9}$$
$$C = \begin{bmatrix} C_y \\ C_w \end{bmatrix} = \begin{bmatrix} \frac{1}{R_C} & 0 \\ -\frac{1}{R_C} & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10}$$

(b) Compute the transfer function of each system, and hence determine the resulting I/O differential equations relating u, y, on the one hand, and u, w on the other.

Solution. The transfer function is:

$$\mathbf{H}(s) = \begin{bmatrix} \mathbf{H}_y(s) \\ \mathbf{H}_w(s) \end{bmatrix} = C(s\mathbb{I} - A)^{-1}B + D \tag{11}$$

$$= \begin{bmatrix} \frac{1}{R_C} & 0 \\ -\frac{1}{R_C} & 1 \end{bmatrix} \begin{bmatrix} s + \frac{1}{CR_C} & -\frac{1}{C} \\ \frac{1}{L} & s + \frac{R_L}{L} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} \tag{12}$$

$$= \begin{bmatrix} \frac{1}{R_C} & 0 \\ -\frac{1}{R_C} & 1 \end{bmatrix} \frac{1}{s^2 + s\frac{L+R_C R_L C}{LCR_C} + \frac{R_L+R_C}{LCR_C}} \begin{bmatrix} s + \frac{R_L}{L} & \frac{1}{C} \\ -\frac{1}{L} & s + \frac{1}{CR_C} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} \tag{13}$$

$$= \frac{1}{s^2 + s\frac{L+R_C R_L C}{LCR_C} + \frac{R_L+R_C}{LCR_C}} \begin{bmatrix} \frac{1}{R_C} & 0 \\ -\frac{1}{R_C} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{LC} \\ \frac{1}{L}(s + \frac{1}{CR_C}) \end{bmatrix} \tag{14}$$

$$= \frac{1}{s^2 + s\frac{L+R_C R_L C}{LCR_C} + \frac{R_L+R_C}{LCR_C}} \begin{bmatrix} \frac{1}{LCR_C} \\ \frac{1}{L}s \end{bmatrix} \tag{15}$$

$$= \begin{bmatrix} \dfrac{\frac{1}{LCR_C}}{s^2 + s\frac{L+R_C R_L C}{LCR_C} + \frac{R_L+R_C}{LCR_C}} \\ \\ \dfrac{\frac{1}{L}s}{s^2 + s\frac{L+R_C R_L C}{LCR_C} + \frac{R_L+R_C}{LCR_C}} \end{bmatrix} \tag{16}$$

From here, the I/O differential equations of for Σ_y and Σ_w respectively are:

$$LCR_C y^{(2)} + (R_L R_C C + L)\dot{y} + (R_L + R_C)y = u \tag{17}$$

$$LCR_C w^{(2)} + (R_L R_C C + L)\dot{w} + (R_L + R_C)w = CR_C\dot{u} \tag{18}$$

For questions **(c)** – **(e)** *assume* $C\text{ap} = 2F$, $R_C = 1\Omega$, $R_L = \frac{5}{3}\Omega$, $L = \frac{2}{3}H$,

(c) Determine the poles λ_i and the zeros z_i for each system. Are the two systems BIBO stable? Explain. Compute the matrix exponential e^{At}.

Solution. The transfer function has the form:

$$\mathbf{H}(s) = \begin{bmatrix} \frac{p_1(s)}{q(s)} \\ \frac{p_2(s)}{q(s)} \end{bmatrix} \tag{19}$$

So the poles for both Σ_y and Σ_w are the roots of $q(s)$. Using (16) with the given values:

$$q(s) = s^2 + 3s + 2 \Rightarrow \lambda_1 = -2,\ \lambda_2 = -1 \Rightarrow \text{BIBO stability}$$

$$p_1(s) = \frac{3}{4} \Rightarrow z_w = \infty, \quad \text{since } \lim_{s\to\infty} \mathbf{H}_w(s) = 0$$

$$p_2(s) = \frac{3}{2}s \Rightarrow z_y = 0$$

Next, we compute the matrix exponential:

$$e^{At} = \mathcal{L}^{-1}[(s\mathbb{I} - A)^{-1}] \tag{20}$$

$$= \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)} \begin{pmatrix} s + \frac{5}{2} & \frac{1}{2} \\ -\frac{3}{2} & s + \frac{1}{2} \end{pmatrix}\right] \tag{21}$$

$$= \mathcal{L}^{-1} \begin{bmatrix} \frac{-1/2}{s+2} + \frac{3/2}{s+1} & \frac{-1/2}{s+2} + \frac{1/2}{s+1} \\ \frac{3/2}{s+2} + \frac{-3/2}{s+1} & \frac{3/2}{s+2} + \frac{-1/2}{s+1} \end{bmatrix} \tag{22}$$

$$= \begin{bmatrix} -\frac{1}{2}e^{-2t} + \frac{3}{2}e^{-t} & -\frac{1}{2}e^{-2t} + \frac{1}{2}e^{-t} \\ \frac{3}{2}e^{-2t} - \frac{3}{2}e^{-t} & \frac{3}{2}e^{-2t} - \frac{1}{2}e^{-t} \end{bmatrix} \mathbf{1}(t) \tag{23}$$

(d) Compute the zero input response of the state $x_{zi}(t)$ as a function of the initial conditions. Compute also the I/O step response for each system Σ_y, Σ_w and identify the steady-state and transient parts.
Solution.

$$x_{zi}(t) = e^{At}\begin{bmatrix} x_1(0^-) \\ x_2(0^-) \end{bmatrix} \tag{24}$$

$$= \begin{bmatrix} (\frac{1}{2}x_1(0^-) + \frac{3}{2}x_2(0^-))e^{-t} + (-\frac{1}{2}x_1(0^-) - \frac{1}{2}x_2(0^-))e^{-2t} \\ (-\frac{3}{2}x_1(0^-) - \frac{1}{2}x_2(0^-))e^{-t} + (\frac{3}{2}x_1(0^-) + \frac{3}{2}x_2(0^-))e^{-2t} \end{bmatrix} \tag{25}$$

The Laplace transform of the step response is given by:

$$\mathbf{R}(s) = \begin{bmatrix} \mathbf{R}_y(s) \\ \mathbf{R}_w(s) \end{bmatrix} = \begin{bmatrix} \mathbf{H}_y(s)\frac{1}{s} \\ \mathbf{H}_w(s)\frac{1}{s} \end{bmatrix} \tag{26}$$

$$= \begin{bmatrix} \frac{3/4}{(s+1)(s+2)s} \\ \frac{3/2s}{(s+1)(s+2)s} \end{bmatrix} \tag{27}$$

$$= \begin{bmatrix} \frac{3/8}{s} + \frac{3/8}{s+2} - \frac{3/4}{s+1} \\ -\frac{3/2}{s+2} + \frac{3/2}{s+1} \end{bmatrix} \Rightarrow \tag{28}$$

$$r(t) = \begin{bmatrix} r_y(t) \\ r_w(t) \end{bmatrix} = \begin{bmatrix} \frac{3}{8} + \frac{3}{8}e^{-2t} - \frac{3}{4}e^{-t} \\ -\frac{3}{2}e^{-2t} + \frac{3}{2}e^{-t} \end{bmatrix} \mathbf{1}(t) \tag{29}$$

(e) In **(d)**, determine a set of initial conditions at $t = 0$ so that for $u(t) = 1$, $t > 0$, the output $y(t)$, $t \geq 0$, contains no transient part.
Solution. The full response is $y(t) = C_y e^{At}x(0^-) + r(t)$. Using (25),

$$Ce^{At}x(0^-) = -\frac{1}{2}e^{-2t}(x_1(0^-) + x_2(0^-)) + e^{-t}(\frac{3}{2}x_1(0^-) + \frac{1}{2}x_2(0^-))$$

so to have no transient components in $y(t)$, using $r_y(t)$, we set:

$$\begin{cases} \frac{3}{2}x_1(0^-) + \frac{1}{2}x_2(0^-) = \frac{3}{4} \\ -\frac{1}{2}x_1(0^-) - \frac{1}{2}x_2(0^-) = -\frac{3}{8} \end{cases} \tag{30}$$

From here we have $x_1(0^-) = x_2(0^-) = \frac{3}{8}$.

Problem [2]. (f) Compute the SVD of the matrices $R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \end{pmatrix}$; in particular compute the singular values, the left and right singular vectors, of R and S as well as the dyadic decomposition of S. What is the best rank one approximation of S and what is the associated error (in the usual 2 norm)?

Solution. Since $RR^* = \text{diag}\,(2,\ 1)$, it follows that $U = I_2$ and the singular values are $\sqrt{2}$, 1. Moreover R has orthogonal columns so V is such that its first and second columns are the same as the first and second rows of R (normalized if necessary); the third column of V is any vector orthonormal to the first two columns. Therefore, we get the following Singular Value Decomposition for R.

$$R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}}_{V^*}$$

Please note the dimensions of the matrices. Σ must always be the same size as R, while U and V are square matrices with dimensions equal to the number of rows of R and the number of columns of R respectively.

Concerning S we notice that if we eliminate the second row and the second column we get the 2×2 matrix $\hat{S} = \begin{pmatrix} 3 & -1 \\ 1 & -3 \end{pmatrix}$.

The block $\begin{pmatrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{pmatrix}$ contributes with a singular value equal to 1, $\sigma_3 = 1$, and the left and right singular vectors $u_3 = v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Therefore we now need to compute the remaining singular values and the associated singular vectors.

We have that $\hat{S}\hat{S}^* = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$ with eigenvalues $10+6 = 16$ and $10-6 = 4$, so the remaining singular values are the square roots of the previous values: $\sigma_1 = 4$ and $\sigma_2 = 2$. It is easy to notice that the eigenvectors of $\hat{S}\hat{S}^*$ are $\hat{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\hat{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so by normalizing and extending the vectors to 3×1 vectors, we get the left singular vectors $u_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $u_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

We have that $\hat{S}^*\hat{S} = \begin{pmatrix} 10 & -6 \\ -6 & 10 \end{pmatrix}$ with the same eigenvalues as $\hat{S}\hat{S}^*$. It is easy to notice that the eigenvectors of $\hat{S}^*\hat{S}$ are $\hat{v}_1 = \hat{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\hat{v}_2 = \hat{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so by normalizing and extending the vectors to 3×1 vectors, we get the left singular vectors $v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ and $v_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

Therefore, we get the following Singular Value Decomposition for S.

$$S = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}}_{V^*}$$

The dyadic decomposition of S is then

$$S = 4 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} + 2 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$$

The best rank one approximation of S is

$$S_1 = 4 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{pmatrix}$$

and the associated error is the next singular value, namely $\sigma_2 = 2$.

Problem [3]. Consider the system given by the I/O differential equation $q(\frac{d}{dt})y = p(\frac{d}{dt})u$, where the polynomials q, p are: $q(s) = (s+\alpha)(s+2)$ and $p(s) = s$. In this problem $\alpha \in \mathbb{R}$ is a real parameter.

(g) Define state variables and write I/S/O equations. For $u(t) = \mathbb{I}(t)$ (step), find $x(t)$ and $y(t)$, *for all* different values of α. (Hint: the form of the solution changes as the parameter takes different values. Therefore the issue is to determine the different forms of the solution as a function of α.)

Solution. A realization is given by

$$A = \begin{pmatrix} 0 & 1 \\ -2\alpha & -(\alpha+2) \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

The step response of the state is $X(s) = (sI - A)^{-1}B\frac{1}{s} = \begin{pmatrix} \frac{1}{(s+\alpha)(s+2)} \\ \frac{1}{s(s+\alpha)(s+2)} \end{pmatrix}$, while the step response of the output is $Y(s) = H(s)\frac{1}{s} = \frac{1}{(s+2)(s+\alpha)}$. Three cases must be distinguished, namely, (i) $\alpha = 0$, (ii) $\alpha = 2$, and (iii) $\alpha \neq 0, 2$:

$$\text{(i)} \quad x(t) = \begin{pmatrix} \frac{1}{2} - \frac{1}{2}e^{-2t} \\ -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t} \end{pmatrix} \mathbb{I}(t), \qquad y(t) = \left(\frac{1}{2} - \frac{1}{2}e^{-2t}\right)\mathbb{I}(t)$$

$$\text{(ii)} \quad x(t) = \begin{pmatrix} te^{-2t} \\ \frac{1}{4} - \frac{1}{4}e^{-2t} - \frac{1}{2}te^{-2t} \end{pmatrix} \mathbb{I}(t), \qquad y(t) = te^{-2t}\mathbb{I}(t)$$

$$\text{(iii)} \quad x(t) = \begin{pmatrix} \frac{1}{\alpha-2}\left(e^{-2t} - e^{-\alpha t}\right) \\ \frac{1}{\alpha-2}\left(-2e^{-2t} + \alpha e^{-\alpha t}\right) \end{pmatrix} \mathbb{I}(t), \quad y(t) = \frac{1}{\alpha-2}\left(e^{-2t} - e^{-\alpha t}\right)\mathbb{I}(t)$$

Notice that in cases (i) and (ii) there are double poles, and hence the form of the solution changes. The critical values $0, 2$ can also be deduced from (iii), as this expression is undefined for these values of the parameters.

(h) Find the value(s) of α for which the impulse response $h(t)$ has a steady state part. Finally, for $u = 0$, find the values of α and values of the initial conditions for which $y(t)$ has no transient part.

Solution. $h(t)$ has a steady-state part equal to 0 for $\alpha \geq 0$. For $\alpha < 0$, the impulse response contains a non-decaying exponential $e^{-\alpha t}$ so the steady-state response does not exist.

For $u = 0$, we have that the response is $y(t) = Ce^{At}x(0^-)$. The eigenvalues of A are $\lambda_1 = -2$ and $\lambda_2 = -\alpha$ so e^{At} will have contain the exponentials e^{-2t} and $e^{-\alpha t}$, which are both transient, unless $\alpha = 0$. In this case, we need to choose the initial conditions $x(0^-)$ in the span of the eigenvector corresponding to the eigenvalue $\lambda = 0$ (i.e., we need to choose $x(0^-)$ in the kernel of A). This is $x(0^-) = k\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where $k \in \mathbb{R}$.

Problem [4]. Given is the system Σ defined by $\dot{x}(t) = Ax(t) + Bu(t)$, where $x(t) = [x_1(t),\ x_2(t),\ x_3(t)]^T$, is the state, $u(t) = [u_1(t),\ u_2(t)]^T$, is the input and

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

We also define the systems Σ_i, obtained from Σ by letting $u_i(t) = 0$, for $i = 1, 2$.

(i) Determine the controllability of each one of these three systems; in particular determine the dimensions and bases for the controllability spaces of Σ, Σ_1, Σ_2. Finally, for system Σ_2, can the state

$$[3/\sqrt{2},\ -1/\sqrt{2},\ -2/\sqrt{2}]^T$$

be steered to zero in $T = 11.5$ seconds? Justify your answers.

Solution. Controllability for Σ:

$$R_3(A,B) = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 2 & 0 & 4 \\ -1 & 1 & -1 & 2 & 0 & 4 \end{bmatrix}$$

The controllability matrix has rank 3 (check that the first 3 columns are indeed linearly independent by computing the determinant of the matrix resulting from considering the first 3 columns only: $\det R_3(A,B)(:,1:3) = -4$) so Σ is completely controllable. A basis for the controllable space is any basis in $\mathbb{R}^3$.

Controllability for Σ_1 - choose the 2^{nd}, 4^{th} and 6^{th} columns of R_3:

$$R_3(A,B_2) = \begin{bmatrix} B_2 & AB_2 & A^2B_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \end{bmatrix}$$

The controllability matrix has rank 1 (all the columns are a multiple of the first one) so Σ_1 is not completely controllable. A basis for the controllable space is the first column.

$$X^{contr} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Controllability for Σ_2 - choose the 1^{st}, 3^{rd} and 5^{th} columns of R_3:

$$R_3(A,B_1) = \begin{bmatrix} B_1 & AB_1 & A^2B_1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

The controllability matrix has rank 2 (check that the first 2 columns are indeed linearly independent by computing the determinant of the matrix resulting from considering the 2×2 block $R_3(A,B_1)(1:2,1:2)$: $\det R_3(A,B_1)(1:2,1:2) = 1$) so Σ_2 is not completely controllable. A basis for the controllable space is

$$X^{contr} = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -2 \end{bmatrix} \right\}$$

The state

$$[3/\sqrt{2},\ -1/\sqrt{2},\ -2/\sqrt{2}]^T$$

cannot be steered to zero because it is not in the controllable space. Check that by assuming that it can be expressed as a linear combination of the basis vectors

$$\begin{bmatrix} \frac{3}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{2}} \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

We have that $a = \frac{3}{\sqrt{2}}$, $b = -\frac{1}{\sqrt{2}}$ but $-a - 2b = -\frac{1}{\sqrt{2}} \neq -\frac{2}{\sqrt{2}}$.