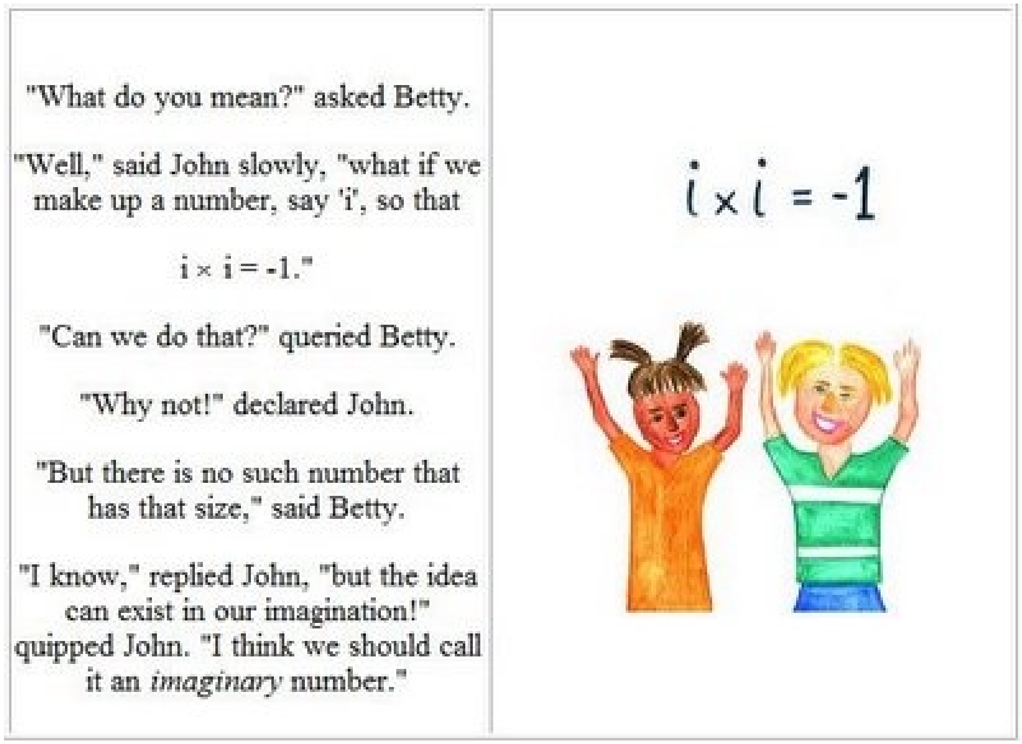
A review of complex arithmetic



"What do you mean?" asked Betty.

"Well," said John slowly, "what if we make up a number, say 'i', so that

i × i = -1."

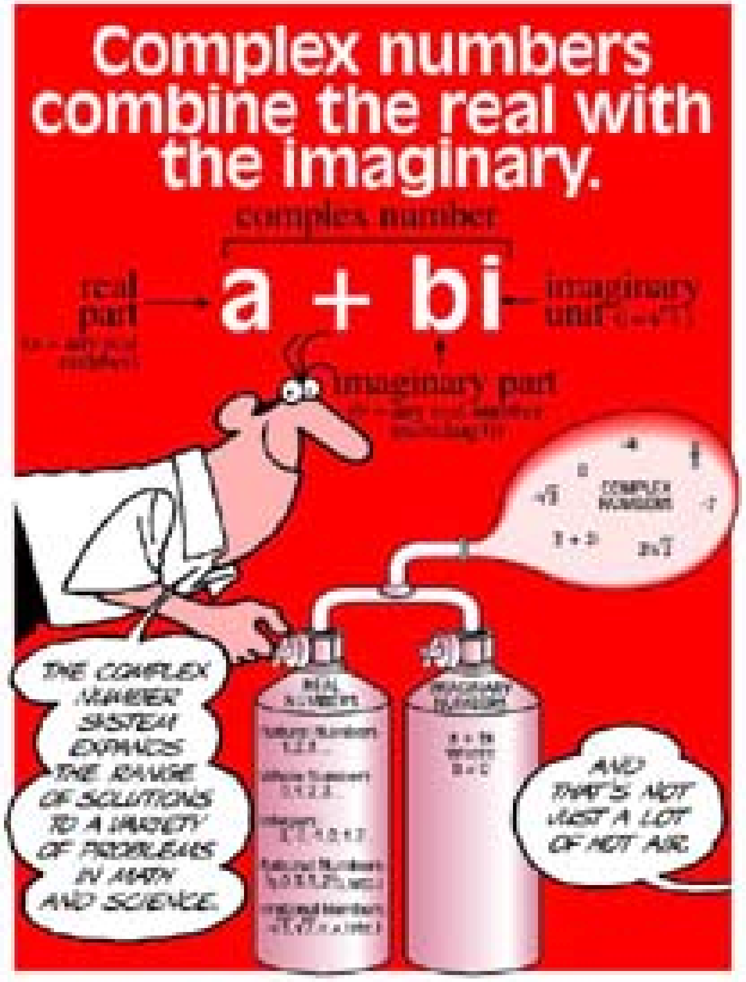
"Can we do that?" queried Betty.

"Why not!" declared John.

"But there is no such number that has that size," said Betty.

"I know," replied John, "but the idea can exist in our imagination!" quipped John. "I think we should call it an *imaginary* number."

From: “John and Betty’s journey into complex numbers”: http://mathforum.org/~mbower/johnandbetty/



The fundamental complex number

There is no number that, when multiplied by itself, gives -1.
So let's define one. We'll call it "*j*".

$$\boxed{j = \sqrt{-1}}$$

<u>The controversy:</u> ***i* vs *j***

In physics: $i = \sqrt{-1}$

and *j* = current density

In engineering: $j = \sqrt{-1}$

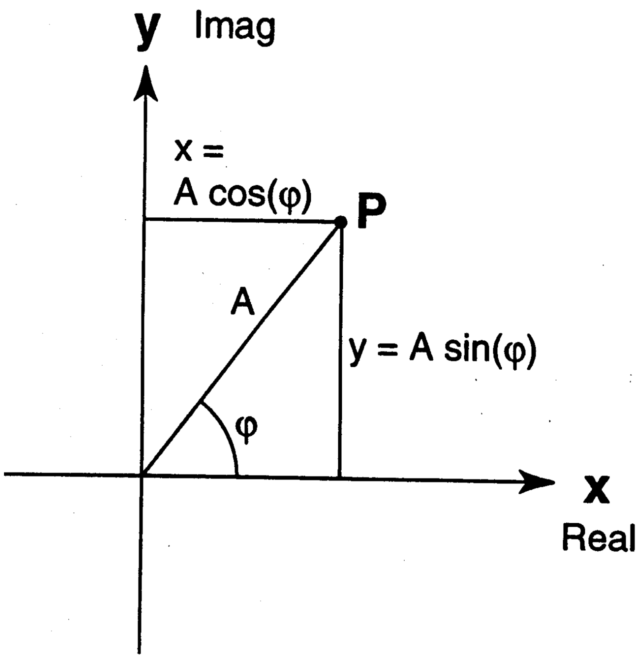
and *i* = current

In this class:

Could be either! But usually we'll use *j*.

(what a pain…but we won't see current too often…)

The complex plane



Consider a point, P = (x,y), on a 2D Cartesian grid.

Let the x-coordinate be the real part and the y-coordinate the imaginary part of a complex number.

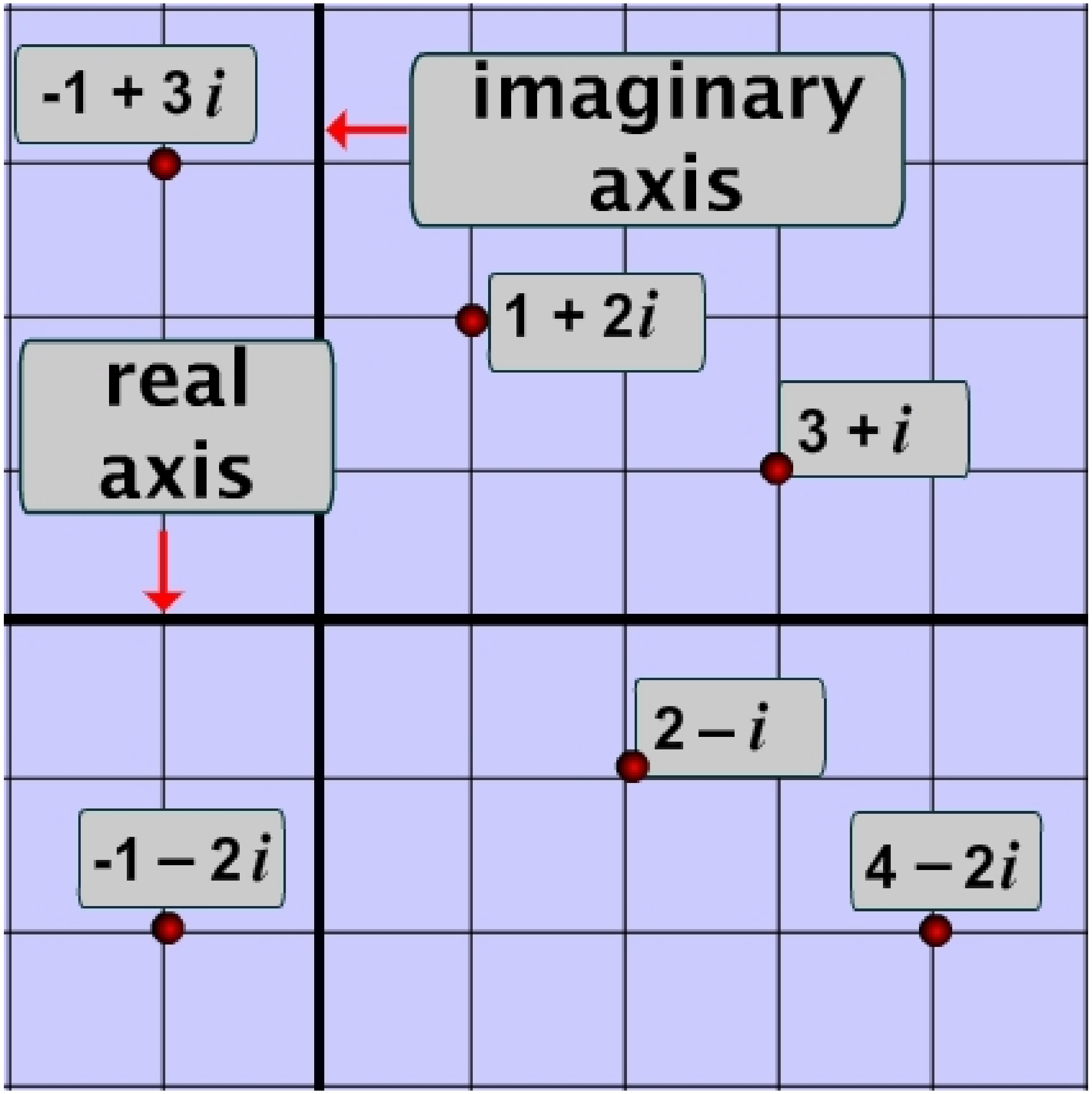
So, instead of using an ordered pair, (x,y), we write:

$$\begin{aligned} P &= x + j\,y \\ &= A\cos(\varphi) + j\,A\sin(\varphi) \end{aligned}$$

In this way, we can represent any complex number as a point on a 2D cartesian plane: the ‘complex plane’.

The complex plane - examples

A complex number contains twice as much information as a real number, because it has two numbers associated with it.



Euler's Formula

$$\boxed{e^{j\phi} = \cos\phi + j\sin\phi}$$

Truly one of the oddest equations in all the world…

so the point, $P = \mathrm{A}\cos(\varphi) + j\,A\sin(\varphi)$, can be written:

$$P = A\exp(j\,\varphi)$$

where: A = Amplitude

φ = Phase

Proof of Euler's Formula $e^{j\phi} = \cos\phi + j\sin\phi$

Use Taylor Series: $f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots$$

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots$$

If we substitute $x = j\varphi$ into $\exp(x)$, then:

$$\exp(j\varphi) = 1 + \frac{j\varphi}{1!} - \frac{\varphi^2}{2!} - \frac{j\varphi^3}{3!} + \frac{\varphi^4}{4!} + \ldots$$

$$= \left[1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} + \ldots \right] + j \left[\frac{\varphi}{1!} - \frac{\varphi^3}{3!} + \ldots \right]$$

$$= \cos(\varphi) + j\sin(\varphi)$$

One consequence of Euler's formula

$$\exp(j\pi) = \cos(\pi) - j\sin(\pi)$$
$$= -1$$

That's a truly strange result. It says:

> "If we multiply 2.718 by itself 3.14 times, and then multiply that result by itself *j* times (whatever that means), we get –1."

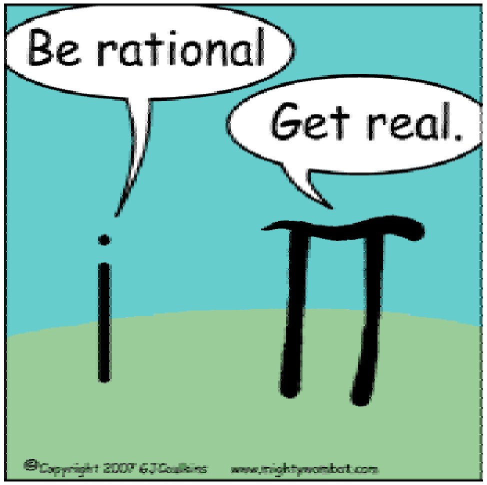
To make it less strange, think of the complex plane again.

P = (x,y) = x + *j* y

The x component is cos(π) = –1.
The y component is sin(π) = 0.

So P = –1 + 0**j*

Also, $\exp(j\pi/2) = j$



Real and imaginary parts

If z is a complex number, $z = x + j\,y$,

Then the 'real part' of the complex number is written:

$$\mathrm{Re}\{ z \} = x$$

and the 'imaginary part' is written:

$$\mathrm{Im}\{ z \} = y$$

NOTE: the real and imaginary parts of z are always REAL numbers.

Complex number theorems

If $\exp(j\varphi) = \cos(\varphi) + j\sin(\varphi)$

Then: $$\exp(-j\varphi) = \cos(\varphi) - j\sin(\varphi)$$

$$\cos(\varphi) = \frac{1}{2}\left[\exp(j\varphi) + \exp(-j\varphi)\right]$$

$$\sin(\varphi) = \frac{1}{2j}\left[\exp(j\varphi) - \exp(-j\varphi)\right]$$

$$A_1\exp(j\varphi_1) \times A_2\exp(j\varphi_2) = A_1A_2\exp\left[j(\varphi_1 + \varphi_2)\right]$$

$$A_1\exp(j\varphi_1) / A_2\exp(j\varphi_2) = A_1 / A_2\exp\left[j(\varphi_1 - \varphi_2)\right]$$

Complex conjugation

We are often interested in the ‘conjugate’ of a complex number.

If z is a complex number, $z = x + j\, y$,

then its ‘complex conjugate is $z^* = x - j\, y$.

The star on z^* tells us that it is the complex conjugate of z.

This simply means: reverse the sign of the imaginary part.

More complex number theorems

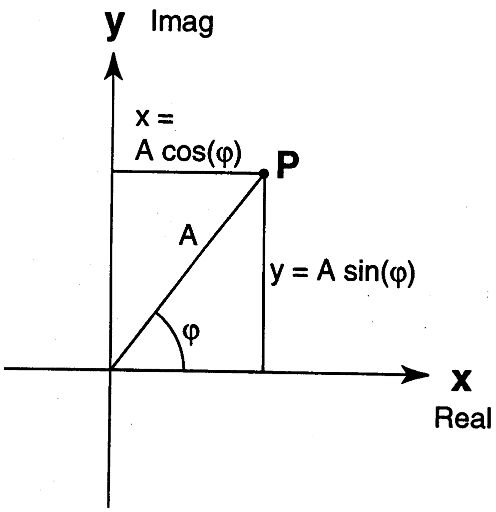
Any complex number, z, can be written in two equivalent ways:

Cartesian form: Polar form:

$$z = \mathrm{Re}\{ z \} + j\, \mathrm{Im}\{ z \} \quad \text{or} \quad z = \mathrm{A} \exp\{ j\, \phi \}$$

So $$\mathrm{Re}\{ z \} = 1/2 \; (z + z^*)$$

and $$\mathrm{Im}\{ z \} = 1/2j \; (z - z^*)$$



The "magnitude," $| z |$, of a complex number is:

$$| z |^2 = z\, z^* = \mathrm{Re}\{ z \}^2 + \mathrm{Im}\{ z \}^2$$

To convert z into polar form, A exp{ $j\, \phi$ } :

$$A^2 = \mathrm{Re}\{ z \}^2 + \mathrm{Im}\{ z \}^2$$

$$\tan(\varphi) = \mathrm{Im}\{ z \} \,/\, \mathrm{Re}\{ z \}$$

This is really important! We often need to convert back and forth between the two equivalent ways to write a complex number.

We can also differentiate exp(jkx) as if the argument were real.

$$\frac{d}{dx}\exp(jkx) = jk \cdot \exp(jkx)$$

Proof :

$$\frac{d}{dx}\left[\cos(kx) + j\sin(kx)\right] = -k\sin(kx) + jk\cos(kx)$$

$$= jk\left[-\frac{1}{j}\sin(kx) + \cos(kx)\right]$$

But $-1/j = j$, so :

$$= jk\left[j\sin(kx) + \cos(kx)\right]$$

QED

In most cases, numbers with *j* in them can be treated as simply numbers, as long as you keep track of the *j*.

Today's lecture was brought to you

by the letter ~~i~~ j