On Design Criteria and Construction of Non-coherent Space-Time Constellations

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Abstract

We consider the problem of digital communication in a Rayleigh flat fading environment using a multiple antenna system, when both the transmitter and the receiver are unaware of the channel coefficients. Using Stein's lemma to bound the pairwise error probabilities, we propose to use the Kullback-Leibler distance between distributions assigned to the transmitted symbols to design space-time constellations for non-coherent communication. We show that optimal codes according to the proposed criterion can be obtained by partitioning the signal space into appropriate subsets and using unitary designs inside each subset. Actual constellation design is performed by solving an optimization problem to find the size and location of the subsets. Our simulation results show that the new constellations can provide a substantial improvement in the performance over known unitary space-time codes. Furthermore, the proposed constellations can be decoded using a simple low-complexity decoder.

Keywords

Non-coherent detection, space-time codes, multiple antenna systems, fading channels, channel coding, wireless communications

I. INTRODUCTION

Exploiting propagation diversity by using multiple antennas at the transmitter and receiver in wireless communication systems has recently been proposed and studied using different approaches [1–8]. In [1, 2], it has been shown that in a Rayleigh flat-fading environment, the capacity of a multiple antenna system increases linearly with the smaller of the number of the transmit and receive antennas, provided that the fading coefficients are known at the receiver. In a slowly fading channel, where the fading coefficients remain approximately constant for many symbol intervals, the transmitter can send training signals that allow the receiver to accurately estimate the fading coefficients. In this case the results of [1, 2] are applicable.

In fast fading scenarios, however, fading coefficients can change into new, almost independent values before being learned by the receiver through training signals. This problem becomes even more acute when large numbers of transmit and receive antennas are being used by the system, which requires very long training sequences to estimate the fading coefficients. A non-coherent detection scheme, where receiver detects the transmitted symbols without having any information about the current realization of the channel, is more suitable for these fast fading scenarios. The capacity of non-coherent systems has been studied in [5, 6], where it has been shown that at high SNR's, or when the coherence interval, T, is much greater than the number of transmit antennas, M, capacity can be achieved by using a constellation of unitary matrices (i.e. with orthonormal columns). These so called unitary space-time codes have exponential encoding and decoding complexity. In [7] another unitary space-time constellation has been proposed which has polynomial encoding but exponential decoding complexity. Differential coding schemes with simple encoding/decoding algorithms have also been proposed [9] for transmit diversity. In [8], a design technique for unitary space-time codes with simple encoding/decoding algorithms has been proposed.

All of the unitary codes of [6–8] have been designed for the case when $T > 1$. For $T = 1$, they provide only one signal point, which is obviously incapable of transmitting any information. The capacity of discrete-time fast Rayleigh fading channels ($T = 1$) has been studied in [10], where it has been shown that the capacity achieving distribution is discrete with a finite number of points, one of them always located at the origin. It appears that the unitary designs are not completely using the information about the statistics of the fading. It can be easily seen that, e.g., for the case of $M = 1$ and $T = 2$, a non-Bayesian approach (i.e., assuming that fading is unknown, with no information about its distribution), would result in a unitary design. This has been a motivation for us to look for a more practical design criterion, rather than asymptotic results based on capacity analysis. The exact expression or even the Chernoff upper bound for average probability of error for the general non-coherent system appears to be intractable. Instead, we consider pairwise error probabilities, and use Stein's lemma [11] to propose the use of KL distance between distributions to approximate the rate of the exponential decay in pairwise error probabilities. This results in a design criterion which coincides with the unitary designs at very high SNR's or very low rates, but for high rate codes or at low SNR's, it results in different signal sets which show better probability of error performance.

In Section II, we introduce the model for the system being considered throughout this paper. In Section III, we derive the KL distance between distributions assigned to transmitted signals, and propose the design criterion based on that. In Section IV, we solve the optimization problem for designing non-coherent space-time constellations which use unitary designs as part of their structure. Some of the resulting constellations and also performance comparison with known unitary space-time codes are also presented. Finally, we draw some conclusions and directions for future work in Section V.

II. System Model

We consider a communication system with M transmit and N receive antennas in a block Rayleigh flat fading channel with coherence interval of T symbol periods (i.e., we assume that the fading coefficients remain constant during blocks of T consecutive symbol intervals, and change to new, independent values at the end of each block). We use the following complex baseband notation

$$X = SH + W, \tag{1}$$

where S is the $T \times M$ matrix of transmitted signals, X is the $T \times N$ matrix of received signals, H is the $M \times N$ matrix of fading coefficients, and W is the $T \times N$ matrix of the additive received noise. Elements of H and W are assumed to be statistically independent, identically distributed circular complex Gaussian random variables from the distribution $\mathcal{CN}(0,1)$. We intentionally avoid using the scaling factor of $\sqrt{\frac{\rho}{M}}$ of [7] to account for the desired signal to noise ratio (or average power constraint on the constellation). We will see in Section IV, that the structure of the optimal constellation depends on the signal to noise ratio, and optimal constellations of the same size at different SNR's are not necessarily scaled versions of each other. Therefore we capture the SNR factor in the S matrix itself, and use the power constraint $\sum_{t=1}^{T}\sum_{m=1}^{M}\mathbb{E}\left\{|s_{tm}|^2\right\} = P$, where s_{tm}'s are the elements of the signal matrix S.

With the above assumptions, the conditional probability density of the received signal can be written as

$$p(X|S) = \mathbb{E}_H\left\{p(X|S,H)\right\} = \frac{\exp\{-\mathrm{tr}[(I_T + SS^H)^{-1}XX^H]\}}{\pi^{TN}\det^N(I_T + SS^H)}, \tag{2}$$

and assuming a signal set of size L, $\{S_i\}_{i=1}^{L}$, the Maximum Likelihood (ML) detector for this system will have the following form

$$\widehat{S}_{ML} = \arg\max_{l\in\{1,\cdots,L\}} p(X|S_l). \tag{3}$$

If $L = 2$, then the probability of error in ML detection of S_1 (detecting S_2 given that S_1 was transmitted) is given by

$$\Pr(S_1 \to S_2) = \Pr\left\{ \frac{\exp\{-\mathrm{tr}[(I_T + S_2S_2^H)^{-1}XX^H]\}}{\pi^{TN}\det^N(I_T + S_2S_2^H)} > \frac{\exp\{-\mathrm{tr}[(I_T + S_1S_1^H)^{-1}XX^H]\}}{\pi^{TN}\det^N(I_T + S_1S_1^H)} \,\middle|\, S_1 \right\}, \tag{4}$$

and $\Pr(S_2 \to S_1)$ is also given by a similar expression. Now, if we also assume that S_1 and S_2 are transmitted with equal probabilities, then the average probability of error in ML detection will be given by

$$P_e = \frac{1}{2}\Pr(S_1 \to S_2) + \frac{1}{2}\Pr(S_2 \to S_1). \tag{5}$$

For the special case of unitary transmit matrices, i.e., when $S_l^H S_l = (\frac{P}{M})I_M$, the exact expression and Chernoff upper bound for the above probability of error were calculated in [6]. However, the corresponding expressions for the general case of non-unitary matrices are not known.

For $L > 2$, even though (4) is no longer exact, we will use it as an approximation for the pairwise error probability, which will, in turn, be used to derive the design criterion for space-time constellations.

III. Design Criterion

As mentioned in the previous section, the pairwise error probability of non-coherent ML detection is approximately given by (4), and does not appear to admit a simple closed form expression. Therefore, instead of (4), we will use the upper bound on the rate of its exponential decay (by number of independent observations) given by Stein's lemma [11], as our performance criterion. According to Stein's lemma, if $\Pr(S_1 \to S_2)$ is sufficiently small, then $\Pr(S_2 \to S_1)$

decays exponentially in number of independent observations at a rate given by $\mathcal{D}\left(p(X|S_1)||p(X|S_2)\right)$, the Kullback-Leibler (KL) distance [11] between $p(X|S_1)$ and $p(X|S_2)$. Independent observations, in our case, can be obtained by using an outer code which operates over several independent fading intervals, or simply by using multiple receive antennas.

Using (2), the KL distance between $p_i = p(X|S_i)$ and $p_j = p(X|S_j)$ can be calculated as

$$\mathcal{D}(p_i||p_j) = N\,\mathrm{tr}\left\{(I_T + S_iS_i^H)(I_T + S_jS_j^H)^{-1}\right\} - N\ln\det\left\{(I_T + S_iS_i^H)(I_T + S_jS_j^H)^{-1}\right\} - NT. \tag{6}$$

Adopting the KL distance as performance criterion, the signal set design criterion in general will be maximization of the minimum KL distance between distributions assigned to the signal points, i.e., assuming equiprobable signal points,

$$\underset{\frac{1}{L}\sum_{l=1}^{L}||S_l||^2 = P}{\text{maximize}} \quad \min_{i\neq j} \quad \mathcal{D}(p_i||p_j), \tag{7}$$

where $||S_l||^2 = \sum_{t=1}^{T}\sum_{m=1}^{M}|(S_l)_{tm}|^2$ is the total power used to transmit S_l. Since the actual value of N does not affect the maximization in (7), in designing the optimal signal sets we can always assume $N = 1$.

IV. Signal Set Construction

The expression for KL distance in (6) is not very illuminating as it is. Therefore, we study the signal set construction problem through a series of special cases. These special cases can provide an understanding of the nature of the KL distance in (6) by breaking it down into simpler components, which results in a simple systematic code design technique for non-coherent systems.

The only assumption in the last of the four special cases that we consider is that the columns of S are orthogonal, i.e., S can be written as product of a $T\times M$ unitary matrix and an $M\times M$ diagonal matrix with nonnegative diagonal elements. In [5], it has been shown that these signal matrices, if chosen from an appropriate distribution, achieve the capacity of non-coherent system. Thus, from an information theoretic point of view, this is the most general case one should consider. However, since our design criterion is different from maximizing the mutual information, we consider orthogonal design as a special case.

A. Special Case 1: $M = 1$ and $T = 1$

This is the case of single antenna fast fading channels, where signal matrices are complex scalars, and the fading coefficients are independent from one symbol to the other. In this case, the maximin problem of (7) reduces to

$$\underset{\frac{1}{L}\sum_{l=1}^{L}|s_l|^2 = P}{\text{maximize}} \quad \min_{i\neq j} \quad \left\{\mathcal{D}_1(p_i||p_j) = \frac{1+|s_i|^2}{1+|s_j|^2} - \ln\left(\frac{1+|s_i|^2}{1+|s_j|^2}\right) - 1\right\}. \tag{8}$$

In general, $\mathcal{D}_1(p_i||p_j)$ gives the KL distance between constellation points that lie in the same complex plane (see IV-C for more explanation). The following proposition characterizes the solution.

Proposition 1: The solution to the maximin problem (8) is given by $|s_l|^2 = \alpha^{l-1} - 1$, where α is the largest real root of the polynomial $\alpha^L - L(P+1)\alpha + (LP + L - 1)$.

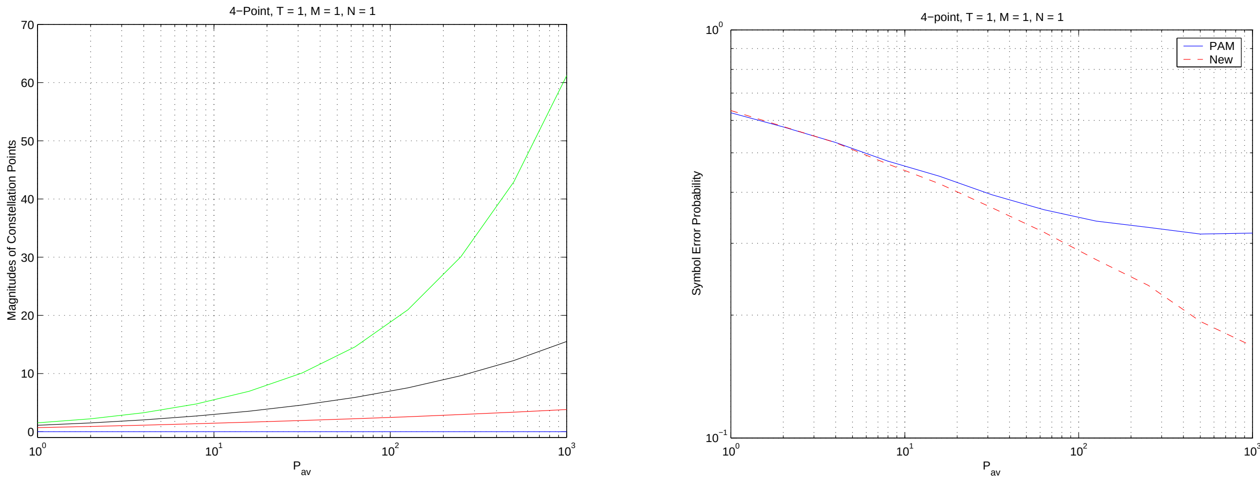
Proof: The proof uses the fact that $f(x) = x - \ln(x) - 1$ is monotonically decreasing for $x \in (0,1)$. The details are omitted due to space limitations. ∎

The resulting constellations are PAM-type constellations but with unequal spacing between the signal points, and the first point is always at zero. The interesting fact is that even the relative locations of the constellation points depend on the SNR, and two constellations of the same size, designed for different SNR's, are not necessarily scaled versions of each other. Figure 1(a) shows the locations of the signal points for a 4-point constellation vs. average transmit power, and Figure 1(b) compares the symbol error rate performance of the optimal constellation with regular PAM constellation at different SNR's.

B. Special Case 2: $M = 1, T > 1$, and $||S_l||^2 = P$ for $l = 1, \cdots, L$

This is the case in which constellation points are column vectors and they all lie on a sphere in $\mathbb{C}^T$. The KL distance of (6) reduces to

$$\mathcal{D}_2(p_i||p_j) = \frac{||S_i||^2||S_j||^2 - |S_i\cdot S_j|^2}{1+||S_j||^2} = \frac{P^2\sin^2(\angle S_i, S_j)}{1+P}, \tag{9}$$



(a) (b)

Fig. 1. (a) Magnitudes of the optimal signal points, and (b) Symbol error rate performance comparison with regular PAM, for a 4-point constellation with $M = 1, T = 1$, and $N = 1$

where $\cdot$ is the inner product operation, and $\angle S_i, S_j$ denotes the angle between signal vectors S_i and S_j. Naturally, this distance depends only on the angle between the signal points. Since all of the points have the same magnitude, this case can be considered as a unitary constellation for $M = 1$. In general, (9) can be used to find the KL distance between constellation points that have the same magnitude.

The optimum constellation in this case, is obviously the one that maximizes the angle between subspaces assigned to the signal points (or equivalently, minimizes the inner product or correlation between signal points). Such designs can be found in [7] and [8].

For $T = 2$, the above criterion results in the signal set $\left\{ \begin{bmatrix} \cos((l-1)\pi/L) \\ \sin((l-1)\pi/L) \end{bmatrix} \right\}_{l=1}^{L}$, which is the same as the signal set proposed in [8]. As also mentioned in [8], these so called PSK constellations have the advantage of low complexity decoding based on a phase comparison. Therefore, we will use these constellations for a more general design explained in the next subsection. Notice that the angle between adjacent points is π/L, not $2\pi/L$. This is because this angle is actually the angle between subspaces containing the constellation points, and thus has to be considered modulo π.

C. *Special Case 3:* $M = 1$ *and* $T \geq 1$

This is the general case of constellations from $\mathbb{C}^T$. They can be considered as block codes for block fading channels where the code word length is equal to the coherence time of the channel. The KL distance in (6) reduces to

$$\begin{aligned}\mathcal{D}(p_i||p_j) &= \underbrace{\frac{1+||S_i||^2}{1+||S_j||^2} - \ln\left(\frac{1+||S_i||^2}{1+||S_j||^2}\right) - 1}_{} + \underbrace{\frac{||S_i||^2||S_j||^2\sin^2(\angle S_i, S_j)}{1+||S_j||^2}}_{} \\ &= \mathcal{D}_1(p_i||p_j) + \frac{||S_i||^2}{||S_j||^2}\mathcal{D}_2(p_i||p_j).\end{aligned} \tag{10}$$

The above partitioning of the KL distance means that the overall distance between two points consists of two parts: $\mathcal{D}_1(p_i||p_j)$ due to having different magnitudes (lying on different spheres in $\mathbb{C}^T$), and $\frac{||S_i||^2}{||S_j||^2}\mathcal{D}_2(p_i||p_j)$ due to the angle between the points (lying on different one-dimensional subspaces of $\mathbb{C}^T$). If two points lie on the same sphere, $\mathcal{D}_1(p_i||p_j) = 0$, and if they lie on the same complex plane (one dimensional subspace), $\mathcal{D}_2(p_i||p_j) = 0$. In general, the overall distance is greater than or equal to either of these parts. This property of the KL distance in (10) suggests partitioning the signal space into subsets of concentric spheres $C_1, \cdots, C_K$, of radii $r_1, \cdots, r_K$, containing $l_1, \cdots, l_K$ points, respectively, and defining the intrasubset and intersubset distances as

$$\mathcal{D}_{intra}(k) = \min_{S_i, S_j \in C_k} \frac{r_k^4 \sin^2(\angle S_i, S_j)}{1+r_k^2} \quad \text{and} \quad \mathcal{D}_{inter}(k, k') = \frac{1+r_k^2}{1+r_{k'}^2} - \ln\left(\frac{1+r_k^2}{1+r_{k'}^2}\right) - 1. \tag{11}$$

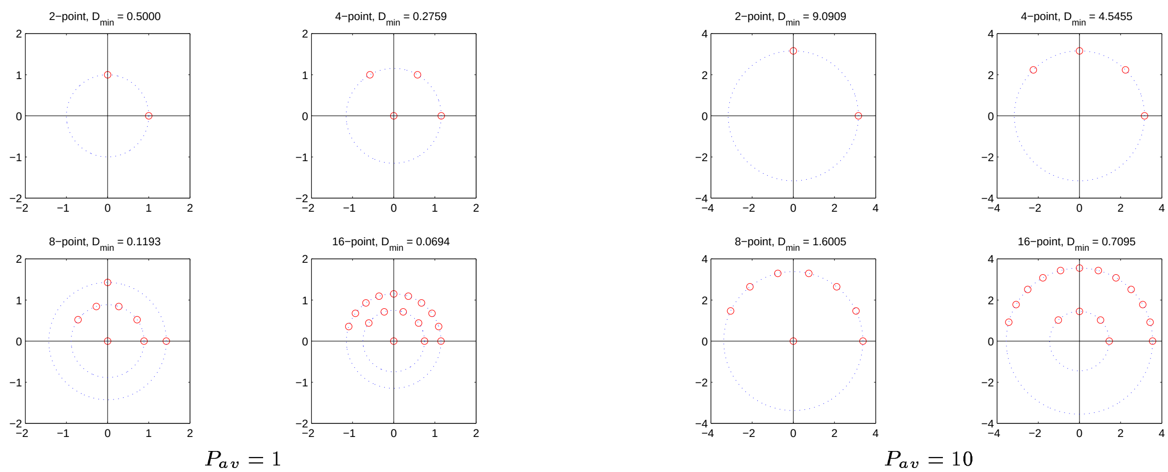


Fig. 2. Optimal constellations of size 2, 4, 8, and 16 for $M = 1, T = 2$

Without loss of generality, we can assume that $r_1 < r_2 < \cdots < r_K$, and reduce the maximin problem in (7) to

$$\underset{\substack{\frac{1}{L}\sum_{k=1}^{K} l_k r_k^2 = P,\ \sum_{k=1}^{K} l_k = L \\ r_1 < r_2 < \cdots < r_K}}{\text{maximize}} \quad \min\left\{\min_{k=1,\cdots,K} \mathcal{D}_{intra}(k), \min_{k=1,\cdots,K-1} \mathcal{D}_{inter}(k,k+1)\right\} . \tag{12}$$

For the special case of $T = 2$, the angle between adjacent points in the kth subset is simply π/l_k, and the maximin problem in (12) can be solved numerically. The resulting 2, 4, 8 and 16-point constellations with average powers of 1 and 10 are shown in Figure 2. As mentioned earlier, each axis in these figures actually represents a complex plane corresponding to one transmit symbol interval. The symbol error rate performance of the 8 and 16-point constellations at $P_{av} = 10$ are simulated for different values of N and compared with the corresponding constellations proposed in [8]. The results are shown in Figure 3. As expected, due to the larger minimum KL distance of the new constellations, the exponential decay of the symbol error rate vs. N is much higher for the new constellations. The minimum KL distances of the new constellations are 1.6005 and 0.7095 for 8-point and 16-point constellations, respectively, whereas the corresponding PSK constellations of [8] have minimum KL distances of 1.3313 and 0.3460, respectively.

The decoding can be done in a similar way to that of trellis coded modulation schemes, i.e., in two phases of "point in subset decoding" and "subset decoding". If a unitary code with low decoding complexity, such as the schemes described in [8], is used inside each subset, then the point in subset decoding phase can be done with a very low cost, and assuming that the number of subsets is much lower than the size of the whole constellation, the overall decoding complexity of the code will be much lower than the regular ML decoder.

D. *Special Case 4: $M \geq 1, T \geq 1$, and $S_l^H S_l = D_l = diag(d_{l1}, \cdots, d_{lM})$ for $l = 1, \cdots, L$*

This is the most general case that we consider in this paper. The last condition (each constellation point is an orthogonal matrix) is used to write $(I_T + S_j S_j^H)^{-1} = I_T - S_j(I_M + D_j)^{-1} S_j^H$, and to simplify (6) to

$$\mathcal{D}(p_i||p_j) = \sum_{m=1}^{M} \left\{ \frac{1 + ||S_{im}||^2}{1 + ||S_{jm}||^2} - \ln\left(\frac{1 + ||S_{im}||^2}{1 + ||S_{jm}||^2}\right) - 1 + \frac{||S_{im}||^2 ||S_{jm}||^2 - \sum_{k=1}^{M} |S_{ik} \cdot S_{jm}|^2}{1 + ||S_{jm}||^2} \right\}, \tag{13}$$

where $||S_{lm}|| = \sqrt{d_{lm}}$ is the magnitude of the mth column of S_l.

The above orthogonality assumption on the signal matrices is equivalent to assuming that $S_l = \Phi_l D_l^{1/2}$, where Φ_l's are unitary matrices. The partitioning of the signal space into subsets of concentric spheres, mentioned in the previous subsection for single transmit schemes, is equivalent to partitioning the signal space into subsets of unitary matrices multiplied by some constant diagonal matrix $D^{1/2}$. Therefore, we can use any existing unitary design as the signal points inside a given subset, and solve the maximin problem in (7) to find the optimum values of D matrices, as well as the number of points in each subset. Here we use the generalized non-coherent unitary PSK design of [8] as the initial unitary design, and construct our design based on that methodology.

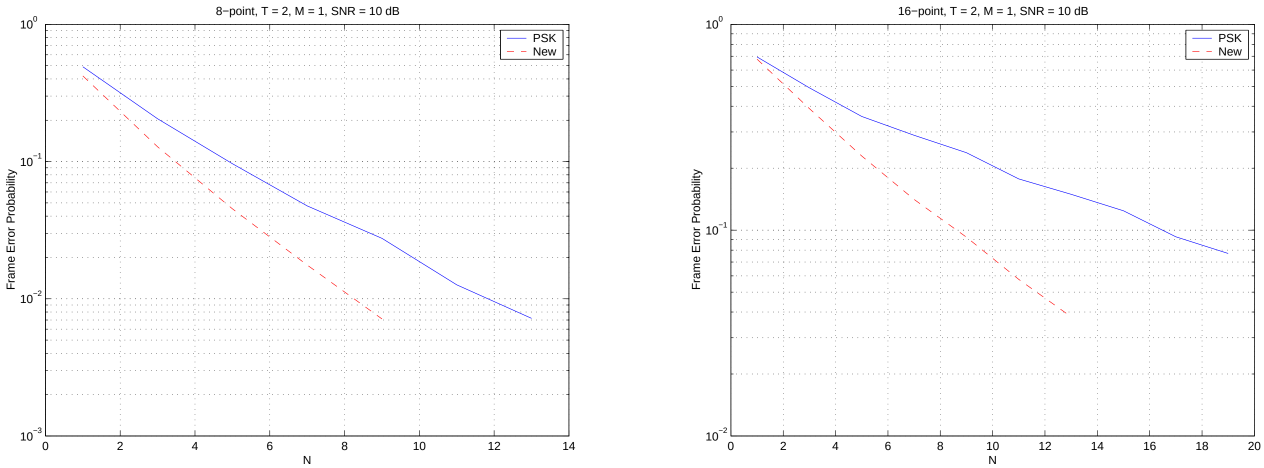


Fig. 3. Symbol error rate for constellations of size 8 and 16 for $M = 1, T = 2$, and SNR = 10 dB

Assuming $S_l = \Phi_l D_l^{1/2}$, where Φ_l's are unitary matrices from the generalized non-coherent PSK constellations of [8], the KL distance in (13) breaks into summation of M KL distances of the form in (10), and each term depends only on one column from each matrix. This means that we can use the construction method of the previous subsection for single transmit antenna, to independently construct constellations for each transmit antenna in this case, and then use the generalized PSK technique to put those vectors together and construct the code matrix. This is, of course, an expected result, because the original generalized PSK scheme is nothing other than designing a PSK constellation for one antenna and transmitting the delayed versions of the same signal through other antennas.

V. Conclusions

We considered the problem of digital communication in a Rayleigh flat fading environment using a multiple antenna system, when neither the transmitter nor the receiver knows the channel coefficients. We derived the design criterion for space-time coding in this scenario based on the Kullback-Leibler distance between distributions assigned to the transmitted symbols. We showed that optimal codes according to the proposed criterion can be obtained by partitioning the signal space into appropriate subsets and using unitary designs inside each subset. We designed new non-coherent constellations and, through simulations, showed that the new constellations can provide a substantial improvement in the performance over known unitary space-time codes. Furthermore, we proposed a low complexity, yet optimal decoding scheme for the designed constellations. Design criteria for the general case of multiple transmit antenna system, as well as the design criteria for an outer code in the case of short coherence intervals, are the major directions for future work.

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